MATH 421. ADVANCED CALCULUS FOR ENGINEERING. FALL 2014. MIDTERM 2

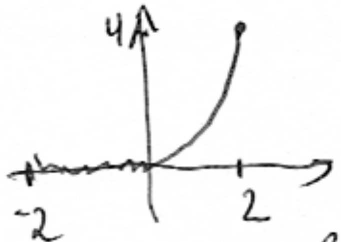
1. (60 points) 1) Find the Fourier series on the interval $-2 < x < 2$ of the function $f(x) = x^2$ for $0 < x < 2$ and $f(x) = 0$ for $-2 < x < 0$.

2) Describe the function which this series represents on the whole line $-\infty < x < \infty$.

3) Specify the numerical series corresponding to the points $x = 0$ and $x = 2$.

4) Let us keep in the Fourier series only terms with Cosines. Which function will represent this new series? Give complete explanations.

$P = 2$



$$a_0 = \frac{1}{2}\int_0^2 x^2\,dx = \frac{8}{6} = \frac{4}{3},$$

$$a_n = \frac{1}{2}\int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right)dx =$$

$$= \underbrace{\frac{1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2}_{=0} - \frac{2}{n\pi}\int_0^2 x \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{4}{(n\pi)^2}\, x\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \frac{4}{(n\pi)^2}\underbrace{\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx}_{=0}$$

$$= \frac{8(-1)^n}{(n\pi)^2}$$

$$b_n = \frac{1}{2}\int_0^2 x^2 \sin\left(\frac{n\pi x}{2}\right)dx = -\frac{1}{n\pi}\, x^2\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 +$$

$$+ \frac{2}{n\pi}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx = \frac{4(-1)^{n+1}}{\pi n} + \underbrace{\frac{4}{(n\pi)^2}\, x\sin\left(\frac{n\pi}{2}\right)\Big|_0^2}_{=0}$$

$$- \frac{4}{n^2\pi^2}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx =$$

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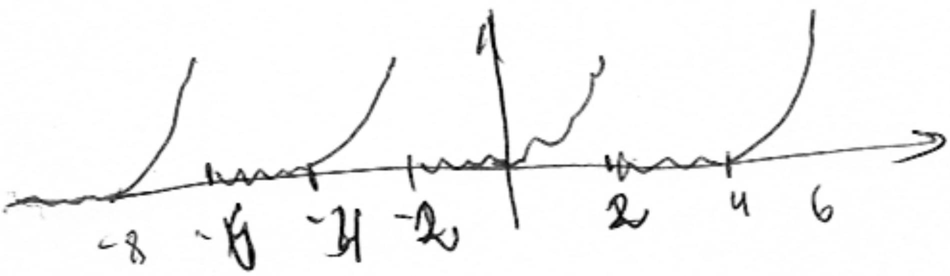
$$= \frac{4(-1)^{n+1}}{\pi n} + \frac{8}{\pi^3 n^3} \cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 =$$

$$= \frac{4(-1)^{n+1}}{\pi n} + \frac{8}{n^3\pi^3}\left((-1)^n - 1\right),$$

$$f(x) = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right) +$$

$$-2 \le x < 2 \quad + \frac{4}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}\pi^2}{n} + \frac{2((-1)^n - 1)}{n^3}\right)\sin\left(\frac{n\pi x}{2}\right)$$

3)



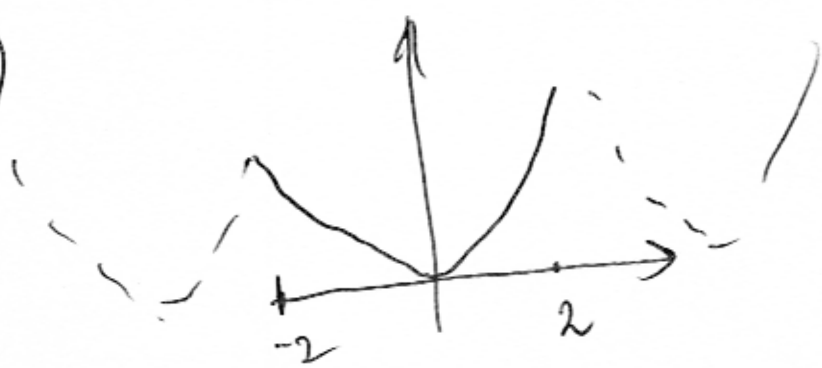
2) $x = 0$

$$0 = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \quad \Big| \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$x = 2$

$$2 = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \quad \Big| \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2},$$

4)



$f_e(x) = \frac{x^2}{2}, \quad -2 \le x \le 2$

2. (45 points) 1) Give the formulas connecting the coefficients of real and complex Fourier series of the same function.

2) Compute the complex Fourier series of the function from Problem 1 on the interval $-2 < x < 2$ by 2 methods:

(i) by the direct computation of the coefficients;

(ii) by the transformation of the coefficients of the real Fourier series from Problem 1 in the coefficients of the complex series.

Compare the results.

1) $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$; $n \geq 0$

$$c_n = \frac{a_n - i b_n}{2}, \quad -\infty < n < \infty; \quad c_0 = \frac{a_0}{2}.$$

2) (ii) $$c_n = \frac{4(-1)^n}{\pi^2 n^2} + i\left(\frac{2(-1)^n}{\pi n} + \frac{4(1-(-1)^n)}{\pi^3 n^3}\right);$$

$$c_0 = \frac{a_0}{2} = \frac{2}{3}.$$

(i) $$c_n = \frac{1}{4}\int_0^2 x^2 \exp\left(-i\frac{n\pi x}{2}\right)dx = \frac{i}{2n\pi} x^2 \exp\left(-\frac{in\pi}{2}x\right)\Big|_0^2$$

$n \neq 0$ $$-\frac{i}{n\pi}\int_0^2 x \exp\left(-i\frac{n\pi x}{2}\right)dx = \frac{2(-1)^n}{\pi n} + \frac{2}{n^2\pi^2} x \exp\left(-\frac{in\pi}{2}x\right)\Big|$$

$$-\frac{2}{n^2\pi^2}\int_0^2 \exp\left(-i\frac{n\pi x}{2}\right)dx =$$

$$= \frac{2(-1)^n}{\pi n} i + \frac{4(-1)^n}{n^2\pi^2} - i\frac{4}{\pi^3 n^3}\exp\left(-\frac{in\pi x}{2}\right)\Big|_0^2 =$$

$$= \frac{2(-1)^n}{\pi n} i + \frac{4(-1)^n}{n^2\pi^2} + \frac{4(1-(-1)^n)}{n^3\pi^3} i$$ The same formula

$c_0 = \frac{1}{4}\int_0^2 x^2 dx = \frac{2}{3}$. $\quad f(x) = \sum_{-\infty}^{\infty} c_n \exp\left(-i\frac{n\pi x}{2}\right)$.

3. (45 points) Find the eigenfunctions of the BVP

$$y'' + \lambda y = 0, y'(0) = 0, y(1) = 0.$$

1) $\lambda > 0$, $\lambda = a^2$, $a > 0$

$y_a(x) = c_1 \cos ax + c_2 \sin ax$,

$y_a'(x) = -ac_1 \sin ax + ac_2 \cos ax$,

$y_a'(0) = 0 \Rightarrow c_2 = 0$, $y_a(x) = c \cos ax$,

$y_a(1) = c \cos a = 0 \Rightarrow a = \frac{\pi}{2}(2k+1)$, $k = 0, 1, 2, \ldots$

$y_k(x) = \cos\left(\frac{\pi}{2}(2k+1)x\right)$ are eigen functions with the eigen ~~functions~~ values $\lambda_k = \frac{\pi^2}{4}(2k+1)^2$, $k = 0, 1, 2, \ldots$

2) $\lambda < 0$, $\lambda = -a^2$, $a > 0$

$y_a(x) = c_1 \cosh ax + c_2 \sinh ax$,

$y_a'(x) = ac_1 \sinh ax + ac_2 \cosh ax$

$y_a'(0) = 0 \Rightarrow c_2 = 0$, $y_a(x) = \cosh ax$

$y_a(1) = 0 \Rightarrow \cosh(a) = 0$, but $\cosh(x) > 0$.

Contradiction! No eigen values $\lambda < 0$.

3) $\lambda = 0$, $y_0 = c_1 x + c_0$

$y_0'(x) = c_1$; $y'(0) = 0 \Rightarrow c_1 = 0$, $y_0(x) = c$

$y_0(1) = 0 \Rightarrow c = 0$, $y \equiv 0$. Contradiction! $\lambda = 0$ isn't an eigen value.

4. (50 points) 1) Give the definition of the regular Sturm-Liouville problem and state Theorem on its eigen values and eigen functions.

2)Consider the BVP

$$y'' + (\cos x)y' + (\lambda \sin x)y = 0, \quad y(\pi/4) = 0, y'(\pi/2) = 0.$$

Put the equation in the self-adjoint (Sturm-Liouville) form and give the orthogonality condition of the eigenfunctions.

Is this problem regular?

2) $\mu' = \mu \cos x$,

$\frac{\mu'}{\mu} = \cos x$

$(\ln \mu)' = \cos x$

$\mu = e^{\sin x}$

$(e^{\sin x} y')' + \lambda \sin x \, e^{\sin x} y = 0$

$r = e^{\sin x}$, $p = \sin x \, e^{\sin x}$, $q = 0$

The eigen functions are orthogonal with the weight $p(x)$; $r > 0$, $p > 0$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$

and BVP is regular

Part 3. Partial Differential Equations (Ch. 13).

- 2 variables
- Equations of 2-nd order.
- Linear equations

Classical equations: [Read Sect. 13.2]

(i) $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ - Heat equation

(ii) $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ - Wave equation (String's equation)

(iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ - Laplace's equation

Special case of solutions.

Solutions with separated variables

Example. $\frac{\partial^2 u}{\partial x^2} = g \frac{\partial u}{\partial y}$ (1)

Let us seek solutions

$$u(x,y) = X(x)\,Y(y);$$ (separated variables)

Substitute at (1):

$$X''Y = g\,X\,Y';$$

Divide on gXY:

$$\frac{X''}{gX} = \frac{Y'}{Y}. \quad (2)$$

The left part depends of x; the right one of y. So it's a constant. Let $X, Y \neq 0$

$$\frac{X''}{gX} = \frac{Y'}{Y} = -\lambda.$$

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Then

$$\begin{array}{l} X'' + g\lambda X = 0 \\ Y' + \lambda Y = 0 \end{array} \quad (3).$$

(3) is equivalent to (1), if the variables are separated.

<u>All</u> such solutions have a form

$$u_\lambda(x,y) = X_\lambda(x)\,Y_\lambda(y)$$

where X_λ, Y_λ are solutions of (3) for some λ.

$$\boxed{Y_\lambda(y) = c\exp(-\lambda y).}$$

(1) $\lambda > 0$, $\lambda = a^2, a > 0$

$$u_a(x,y) = \exp(-a^2 y)\,(c_1 \cos(3ax) + c_2 \sin(3ax));$$

(2) $\lambda < 0$, $\lambda = -b^2, b > 0$

$$u_b(x,y) = \exp(b^2 y)\,(d_1 \cosh(3bx) + d_2 \cosh(3bx));$$

(3) $\lambda = 0$

$$u_0(x,y) = f_0 + f_1 x.$$

(independent of y).

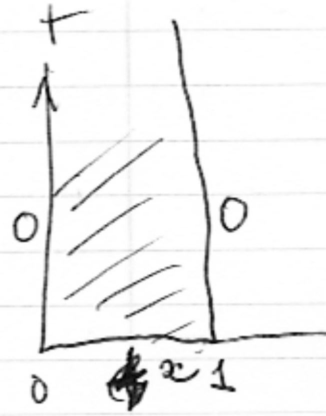
We can take superpositions (linear combinations of these solutions).

BVP for Heat equation (Sect. 13.3)

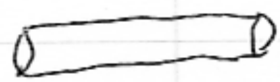
$$(1)\quad 4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1,\ t > 0$$

$$(2)\quad u(0,t) = 0,\ u(1,t) = 0,\ t > 0$$

$$(3)\quad u(x,0) = x, \quad 0 < x < 1$$



Step 1. Solutions with separated variables.



Thin rod of length 1; initial temp. is x; the temp. of the ends is zero.

$$u(x,t) = X(x)\,T(t)$$

$$\frac{X''}{X} = \frac{T'}{4T} = -\lambda.$$

We add the eigen value λ artificialy.

$$\begin{aligned} X'' + \lambda X &= 0 \\ T' + 4\lambda T &= 0 \end{aligned} \quad (4)$$

For separated variables $(4) \Leftrightarrow (1)$.

$$u_\lambda(x,t) = T_\lambda(t)\,X_\lambda(x),$$

$$T_\lambda(t) = \exp(-4\lambda t).$$

We also know possible $X_\lambda(x)$.

Step 2. <u>Boundary conditions (2).</u>

(2) $\Leftrightarrow$ $X_\lambda(0) = X_\lambda(1) = 0.$

$$X_\lambda'' + \lambda X_\lambda = 0$$

So possible eigen functions

$X_n(x) = c_n \sin(n\pi x)$, $n = 1, 2, 3, \ldots$

withe the eigen values $\lambda_n = n^2\pi^2$.

As the result the solutions

$$\boxed{u_n = c_n \exp(-4 n^2 \pi^2 t) \sin n\pi x}$$

satisfy to the equetion (1) and the boundary condition (2). The same for

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-4 n^2 \pi^2 t) \sin(n\pi x).$$

Step 3. Find $\{c_n\}$ such that $u(x,t)$ satisfies (3) as well and as the result to whole BVP (1-3).

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \equiv x, \quad 0 < x < 1$$

We know that

$$x = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x), \quad 0 < x < 1$$

So $c_n = \frac{2}{\pi} \frac{(-1)^{n+1}}{n}$ and

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \exp\left(-4n^2\pi^2 t\right) \frac{(-1)^{n+1}}{n} \sin(n\pi x),$$

is the solution of (1)-(3).

It's possible to show that this solution is unique.

So we found that solutions with separated variables can be used as a base for the solution of BVP (1)-(3).