Lecture 2.

Linear property of the Laplace transform:

$$\boxed{\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L} f_1 + c_2 \mathcal{L} f_2}$$

Exercises:

Find

1. $f(t) = 2t^2 - 3t + 5$

$$\mathcal{L}f(s) = \frac{4}{s^3} - \frac{3}{s^2} + \frac{5}{s}.$$

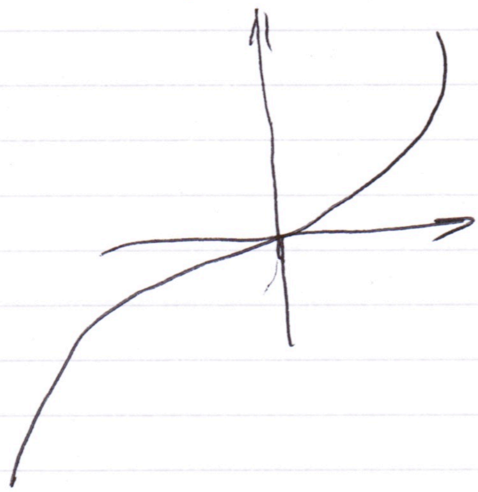
2. $f(t) = -2t^5 + \sin 2t - 3\cos 3t + e^{-2t}$

$$\mathcal{L}f(s) = -\frac{2 \cdot 5!}{s^6} + \frac{2}{s^2+4} - \frac{3s}{s^2+9} + \frac{1}{s+2}$$

Hyperbolic functions.

$\cosh t \overset{\text{déf}}{=} \frac{1}{2}\left(e^{t} + e^{-t}\right),$

$\sinh t = \frac{1}{2}\left(e^{t} - e^{-t}\right)$



$f(t) = \cosh t = \Rightarrow \mathcal{L}f = \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+1}\right) =$

$\underline{\mathcal{L}\{\tfrac{1}{2}(e^{t}+e^{-t})\}}$ $\qquad = \frac{s}{s^2-1}$, $\underline{s>1}$

[If $a>0$ then $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ only for $s>a$]

$g(t) = \sinh t \Rightarrow \mathcal{L}g = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) = \frac{1}{s^2-1}$, $\underline{s>1}$

Table 1

Let us to start a table

$f(t)$	$\mathcal{L}f(s)$
t^n, $n=0,1,\dots$	$\frac{n!}{s^{n+1}}$
e^{at}, $a<0$	$\frac{1}{s-a}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$, $s \geq \lvert k \rvert$
$\cosh kt$	$\frac{s}{s^2-k^2}$, $s \geq \lvert k \rvert$

Lecture 2.

The inverse Laplace transform $\mathcal{L}^{-1}$.

Definition. $f(t) = \mathcal{L}^{-1}F$ iff $F = \mathcal{L}f$.

Examples. Inverse the table!

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{1}{2}t^2$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2-1}\right) = \cosh t$$

We use by the same table but «from right to left»

$$\mathcal{L}^{-1}(aF+bG) = a\mathcal{L}^{-1}F + b\mathcal{L}^{-1}G$$

Examples.

1. $\mathcal{L}^{-1}\left(\frac{3s-1}{s^2+4}\right) = 3\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$

$$= 3\cos 2t - \frac{1}{2}\sin 2t.$$

⇐ Calculus Partial Fractions (undefined coefficients)

Rational fractions: $\frac{P(s)}{Q(s)}$

2. $\mathcal{L}^{-1}\left(\frac{2s}{s^2-3s+2}\right) = \mathcal{L}^{-1}\left(\frac{2s}{(s-2)(s-1)}\right)$

$$\frac{2s}{(s-2)(s-1)} = \frac{a}{s-2} + \frac{B}{s-1} = \frac{(a+B)s-(a+2B)}{(s-2)(s-1)} =$$

$$\left.\begin{array}{l} a+B=2 \\ a+2B=0 \end{array}\right] a=4,\ B=-2 \qquad = \frac{4}{s-2} - \frac{2}{s-1}.$$

$= 4e^{2t} - 2e^{t}$. (Well defined only for $s>2$)

2.4

Transforms of Derivatives

$$\underbrace{(\mathcal{L}f')(s)}= \int_0^\infty \overset{u}{e^{-st}} \overset{v'}{f'(t)}\,dt =$$

$$= e^{-st} f(t)\Big|_0^\infty + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathcal{L}f}$$

$$= \underbrace{-f(0) + s\mathcal{L}f(s).} \qquad \underbrace{\mathcal{L}f' = -f(0) + s\mathcal{L}f}$$

Generalization (without proof).

$$\mathcal{L}f^{(k)}(s) = s^k \mathcal{L}f(s) - \underbrace{s^{k-1}f(0) - s^{k-2}f'(0) - \dots - f^{(k-1)}(0)}_{\text{boundary terms}}.$$

Let us start Table 2.

For the computation of the Laplace transform of derivative we need to know boundary values at $t=0$.

② $\mathcal{L}^{-1}\left(\dfrac{1}{\underbrace{(s+1)(s-2)(s+3)}}\right) = -\frac{1}{6}e^{-t} + \frac{1}{15}e^{2t} + \frac{1}{10}e^{-3t}$

For $s > 2$

$$= \frac{a}{s+1} + \frac{B}{s-2} + \frac{C}{s+3}$$

Numerators

$$a(s-2)(s+3) + B(s+1)(s+3) + C(s+1)(s-2) = 1$$

1st way: Compare coefficients for $s^2, s, 1$, and solve system for a, B, C (3 equations with 3 variables.

2nd way! The substitution.

$s = -1 \Rightarrow -6a = 1 \Rightarrow a = -\frac{1}{6}$

$s = 2 \Rightarrow 15B = 1 \Rightarrow B = \frac{1}{15}$

$s = -3 \Rightarrow 10C = 1 \Rightarrow C = \frac{1}{10}$

$$\begin{array}{l|l} s^2 & a + B + C = 0 \\ s & a + 4B - C = 0 \\ 1 & -6a + 3B - 2C = 1 \end{array}$$

The substitution is impossible sometimes

$(a+B+C)s^2 + (a+4B-C)s + (-6a+3B-2C) = 1$

Table 2.

$$(\mathcal{L} f^{(k)})(s) = s^k \mathcal{L} f(s) - s^{k-1} f(0) - s^{k-2} f'(0) - \dots - f^{(k-1)}(0).$$

Applications to ordinary differential equations.

Examples.

1. $y' - 3y = 2, \; y(0) = 2$ (1)

- $y(x)$ satisfies to the equation
- $y(0) = 2$ (boundary condition)

Step 1. Take the Laplace transform of both parts of the equation.

$\boxed{Y = \mathcal{L}y}$

$$(s-3)Y - y(0) = \frac{2}{s}.$$

s is the variable. We have ~~an~~ an algebraic equation connecting (Y, s):

$$(s-3)Y = \frac{2}{s} + 2 = \frac{2(s+1)}{s},$$

Step 2. $$\boxed{Y(s) = \frac{2(s+1)}{s(s-3)}}$$ We solved the algebraic equation!

Step 3. Take the inverse Laplace transform

$$y(x) = (\mathcal{L}^{-1} Y)(x)$$

Partial fractions.

$$\frac{2(s+1)}{s(s-3)} = 2\left(\frac{a}{s} + \frac{B}{s-3}\right) = \frac{2((a+B)s - 3a)}{s(s-3)}$$

$$\boxed{a = -\frac{1}{3},\ B = \frac{4}{3}}$$

$$a + B = 1$$
$$-3a = 1$$

$$y(x) = \mathcal{L}^{-1}\left(\frac{2}{3}\left(-\frac{1}{s} + \frac{4}{s-3}\right)\right) = \frac{2}{3}\left(-1 + 4e^{3x}\right)$$

Verification: $y(0) = 2$

Problem 2.

$$y' + 2y = \cos 3t, \quad y(0) = -1$$

$y(t)$

Step 1. $Y = \mathcal{L}y$

$$(s+2)Y - y(0) = \frac{s}{s^2+9}$$

$$(s+2)Y = \frac{s}{s^2+9} - 1$$

Step 2. $$Y(s) = -\frac{1}{s+2} + \frac{s}{(s^2+9)(s+2)}$$