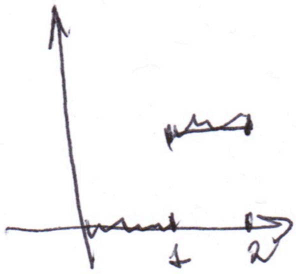
1. (70 points) a)Let the function $f(x) = 1, 1 \le x \le 2$ and $f(x) = 0, 0 \le x \le 1$. Expand it on $[0, 2]$ in the both Fourier cosine and sine series.

b)Describe the sum of these series for all $-\infty < x < \infty$. Give explanations and sketch the graphs.

c)Specify the numerical series which are the results of evaluations for $x = 0, 1$.

d) Take the sum of these cosine and of this Fouries series and consider it as a Fourier series on $[-2, 2]$. Describe the function which will be its sum.

Cosine F. series



a) $p = 2$

$$a_0 = \int_1^2 dx = 1$$

$$a_n = \int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx = \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2 =$$

$$= \frac{2}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) = \begin{cases} 0 & n = 2k \\ \frac{2(-1)^{k+1}}{(2k+1)\pi}, & n = 2k+1 \end{cases}$$

$$\underline{f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\cos\frac{(2k+1)\pi}{2}x, \quad 0<x<2}$$

Sine F. series

$$b_n = \int_1^2 \sin\left(\frac{n\pi}{2}x\right)dx = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\Big|_1^2 = -\frac{2}{n\pi}\left(\cos(n\pi\right.$$

$$- \cos$$

$$= -\frac{2}{n\pi}\left((-1)^n - \begin{cases} 0, & n=2k+1 \\ (-1)^k, & n=2k. \end{cases}\right)$$

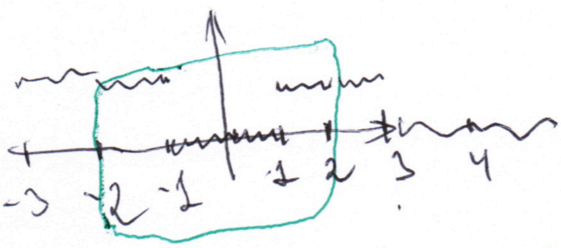
$$f(x) = \frac{2}{\pi}\left(\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{(2k+1)\pi}{2}x\right) + \right.$$

$$\left. + \sum_{l=1}^{\infty}\underline{\frac{(-1)^{l+1}-1}{2l}}\sin\left(l\pi x\right)\right).$$

b), c)

Cosine one

$$x=0;\ 0 = \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$



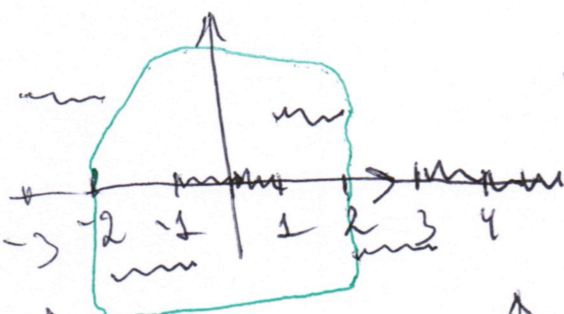
$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

$$x=1 \quad \frac{1}{2} = \frac{1}{2}$$

The

~~Sine on~~

Sine one

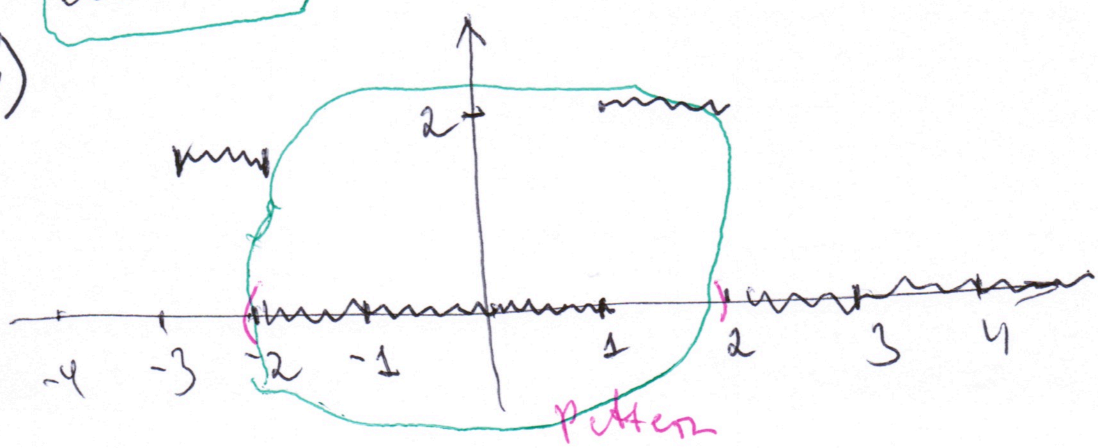


$$x=0, \quad 0=0$$

$$x=1$$

$$\frac{1}{2} = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

d)



4*. (20 extra points). If a function $f(x)$ is even on $[-p, p]$, what can we say about the coefficients of its complex Fourier series? Give a necessary and sufficient condition. Give detailed explanations.

$b_n = i(c_n - c_{-n}) = 0$ for all n,

So $c_n = c_{-n}$, $n = 0, 1, \ldots$

$$c_n = \frac{a_n - i b_n}{2}, \quad n \geq 0$$

$$c_{-n} = \frac{a_n + i b_n}{2}$$

2. (65 points) a) Give Euler's formula.

b) Expand the function $f(x) = e^{\pi x}$ in the complex Fourier series on $[-1, 1]$.

c) Describe the sum of this series for all x.

d)Transform this series in the real Fourier series using the formulas connecting coefficients of real and complex series.

a) $e^{ix} = \cos x + i \sin x$

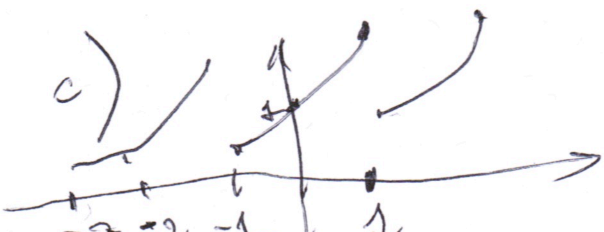
b) $c_n = \frac{1}{2} \int_{-1}^{1} \exp(\pi x - in\pi x)\,dx = \frac{1}{2} \int_{-1}^{1} \exp(\pi x(1-in))\,dx$

$= \frac{1}{2} \frac{1}{\pi(1-in)} \left[\exp(\pi(1-in)) - \exp(-\pi\underbrace{(1-in)}_{(1+in)}) \right]$

$= \frac{(-1)^n}{\pi(1-in)} \sinh(\pi) = \frac{(-1)^n \sinh(\pi)}{\pi(1+n^2)}$ $\quad \frac{e^{in\pi}}{e^{-in\pi}} = (-1)^n$.

$$e^{\pi x} = \frac{\sinh(\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-in} \exp(in\pi x), \quad -1<x<1.$$

c)



$2c_0 = a_0$ $\quad a_0 = a_0$

d) $a_n = c_n + c_{-n} = \frac{(-1)^n}{\pi} \sinh(\pi) \left(\frac{1}{1-in} + \frac{1}{1+in} \right) = \frac{2\sinh\pi\,(-1)^n}{\pi(n^2+1)}$

$b_n = i(c_n - c_{-n}) = \frac{2}{\pi} \sinh(\pi) \frac{(-1)^{n+1} n}{n^2+1}$

$$e^{\pi x} = \frac{2\sinh(\pi)}{\pi} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \left(\cos(n\pi x) + n \sin(n\pi x) \right) \right).$$

$-1 < x < 1$

a new type of Fourier series.

3) $y'' + \lambda y = 0, \quad [0,\pi]$

$y(0) = y'(\pi) = 0 \quad (3)$

1) $\lambda > 0, \ \lambda = a^2, \ a > 0$

$y_a(x) = c_1 \cos ax + c_2 \sin ax.$

$y_a(0) = 0 \Rightarrow c_1 = 0, \ y_a(x) = c \sin ax$ (see above).

$c \neq 0, a \neq 0$

(since $y \equiv 0$ isn't an eigen function)

$y'_a(x) = c\, a \cos ax$

$y'_a(\pi) = 0 \Leftrightarrow \cos a\pi = 0 \Leftrightarrow a\pi = \frac{\pi}{2}(2k+1), \ k = 0, 1, 2 \ldots$

$\boxed{a = \frac{2k+1}{2}}$

$y_k(x) = \sin\left(\frac{2k+1}{2} x\right), \ k = 0, 1, 2, \ldots$ are eigen functions with the eigen values $\lambda_k = \frac{(2k+1)^2}{4}$.

2) $\lambda < 0, \ \lambda = -a^2, \ a > 0$

$y_a(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$

$y_a(0) = 0 \Rightarrow c_1 = 0, \ y_a(x) = c \sinh(ax), \ c \neq 0, a \neq 0$

$y'_a(x) = ca \cosh(ax), \ y'_a(\pi) = 0 \Rightarrow \cosh(a\pi) = 0$ (impossible,

$a > 0$

A new type of Fourier series.

3) $y'' + \lambda y = 0, \quad [0, \pi]$

$y(0) = y'(\pi) = 0$ (3)

1) $\lambda > 0, \ \lambda = a^2, \ a > 0$

$y_a(x) = c_1 \cos ax + c_2 \sin ax$.

$y_a(0) = 0 \Rightarrow c_1 = 0, \ y_a(x) = c \sin ax$ (see above).

$c \neq 0, a \neq 0$

(since $y \equiv 0$ isn't an eigen function)

$y'_a(x) = c\, a \cos ax$

$y'_a(\pi) = 0 \Leftrightarrow \cos a\pi = 0 \Leftrightarrow a\pi = \frac{\pi}{2}(2k+1), \ k = 0, 1, 2, \ldots \Rightarrow$

$\boxed{a = \frac{2k+1}{2}}$

$y_k(x) = \sin\left(\frac{2k+1}{2} x\right), \ k = 0, 1, 2, \ldots$ are eigen functions with the eigen values

$\lambda_k = \frac{(2k+1)^2}{4}$.

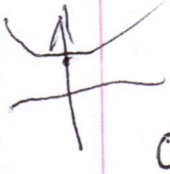
2) $\lambda < 0, \ \lambda = -a^2, \ a > 0$

$y_a(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$

$y_a(0) = 0 \Rightarrow c_1 = 0, \ y_a(x) = c \sinh(ax), \ c \neq 0, \ a \neq 0$

$y'_a(x) = ca \cosh(ax)$, $y'_a(\pi) = 0 \Rightarrow \cosh(a\pi) = 0$ (impossible,

$a > 0$



$\cosh x = \frac{1}{2}(e^{x}+e^{-x}) > 0$, No eigen values $\lambda < 0$.

3) $\lambda = 0$

$y_0(x) = c_1 x + c_0$

$y_0(0) = 0 \Rightarrow c_0 = 0$, $y_0(x) = cx$, $y_0'(x) = c$.

$y_0'(\pi) = 0 \Rightarrow c = 0$, $y_0 \equiv 0$ Contradiction.

$\lambda = 0$ isn't an eigen value.

~~Only~~ $\left\{\sin\left(\frac{2k+1}{2}x\right)\right\}$, $k = 0, 1, 2, \dots$, $\lambda_k = \frac{(2k+1)}{4}$ is the complet system of eigen functions.

Compare with Sine Fourier series

$\{\sin mx\}$

~~A~~ Which boundary problem corresponds to the complet Fourier Series?

$y'' + \lambda y = 0$, $[-\pi, \pi]$

$y(-\pi) = y(\pi)$
$y'(-\pi) = y'(\pi)$ (periodical condition).

1) $\cos(nx)$, $\sin(nx)$, $\lambda_n = n^2$
satisfy to this condition
2 indep. eigen functi

2) $\lambda < 0$
No periodic eigen functions!

3) $\lambda = 0$ $y_0(x) \equiv 1$ is periodic

Examples.

1. Cauchy - Euler eq.

$$x^2 y'' + x y' - \alpha^2 y = 0$$

$$a = x^2, \quad b = x$$

$$\frac{b - a'}{a} = \frac{x - 2x}{x^2} = -\frac{1}{x}$$

$$\mu(x) = \exp\left(-\int \frac{dx}{x}\right) = \exp(-\ln x) = \frac{1}{x}$$

$\Downarrow$

$$\cancel{x} \quad x y'' + y' - \alpha^2 \frac{y}{x} = 0$$

$$(x y')' - \alpha^2 \frac{y}{x} = 0$$

2. Bessel eq.

$$x y'' + y' + \alpha x y = 0$$

$$(x y')' + \alpha x y = 0$$

$$r = x, \; q = 0, \; p = x, \; \lambda = \alpha$$

3. $x^2 y'' + 3 x y' + \lambda y = 0$

Guess: $\mu(x) = x \Rightarrow x^3 y'' + 3x^2 y' + \lambda x y = 0$

$$(x^3 y')' + \lambda x y = 0.$$

$$r = x^3, \; q = 0, \; p = x.$$

$[a, b]$

$a > 0, \; b > 0$

regular

$$\frac{b - a'}{a} = \frac{3x - 2x}{x^2} = \frac{1}{x}$$

$$\mu(x) = \exp\left(\int \frac{dx}{x}\right) = e^{(\ln x)} = x.$$

Lecture 18

Problems from HA.

#9 Laguerre's differential equation

$$xy'' + (1-x)y' + ny = 0$$

$a = x$, $b = (1-x)$ $\qquad \mu = \exp\left(\int \frac{-x}{x}\,dx\right) = e^{-x}$

$L_n(x)$ - polynomial eigen function orthogonal with the weight e^{-x}.

$r = xe^{-x}$, $p = e^{-x}$, $q = 0$

$r > 0, x \neq 0$

$$\boxed{(xe^{-x}y')' + ne^{-x}y = 0}$$

$\lambda = n$

#10. Hermite's differential equation

$$y'' - 2xy' + 2ny = 0$$

$a = 1$, $b = -2x$

$$\mu = \exp\left(\int \frac{b}{a}\,dx\right) = \exp\left(\int(-2x)\,dx\right) = \exp(-x^2)$$

$H_n(x)$ - polynomial system of eigen functions, orthogonal with the weight e^{-x^2}

$$(e^{-x^2}y')' + 2n\,e^{-x^2}y = 0$$

The important application of St.-L. theory -

compet systems of orthogonal polynomials (with weights).

Lagerre, Hermite polynomials.

Choise of the interval $[a, b]$

- remember about $r>0, p>0$.

(no restrictions for Laguerre's & Hermite's polynomials.

The most important example

Legendre polynomials (12.6.2, 12.5 (Ex. 4))

Legendre's equation.

$$\underbrace{(1-x^2)y'' - 2xy'}_{b=a'} + n(n+1)y = 0, [-1, 1]$$

$$((1-x^2)y')' + n(n+1)y = 0$$

$$r = (1-x^2), \quad p \equiv 1, \quad q \equiv 0$$

Eigen functions are orthogonal (in usual sense, $p \equiv 1$)