Lecture 18

Sturm-Liouville Problem. Sect. 12.5

Generalization of the equation $y''+\lambda y=0$.

Sturm-Liouville equation

$$\frac{d}{dx}\left(r(x)\,y'\right)+\left(q(x)+\lambda p(x)\right)y=0 \quad (1)$$

- coefficients $r(x), q(x), p(x)$ (variable) on $[a,b]$
  $r(x)>0$, $p(x)>0$ on $[a,b]$
  (no conditions on $q(x)$).

- Self-adjoint form
  $\frac{d}{dx}(r(x)\,y')=r'y'+r\,y''$
  $ay''+a'y'$ (discuss later).

$y''+\lambda y=0$, $\quad r\equiv1,\ q\equiv0,\ p\equiv1$.

Examples:

1) $y'' + \lambda y = 0$

2) <u>Cauchy - Euler equation.</u>

$$x^2 y'' + x y' - \alpha^2 y = 0, \quad \alpha \geq 0$$

$$\lambda = -\alpha^2$$

General solution:

$$y = c_1 x^{-\alpha} + c_2 x^{\alpha}, \quad \alpha \neq 0$$

$$y = c_1 + c_2 \ln x, \quad \alpha = 0.$$

verify

It's not the self-adjoint form, but we can multiply on $\frac{1}{x}$.

$$x y'' + y' - \alpha^2 \frac{y}{x} = 0$$

$$(x y')' - \alpha^2 \frac{y}{x} = 0$$

$$\underline{r(x) = x, \quad q \equiv 0, \quad p} = \frac{1}{x}, \quad \lambda = -\alpha^2 ?$$

How to transform an arbitrary differential equation to the self-adjoint form?

— By multiplying by an integrating factor $\mu(x)$ (positive).

$$\mu(x)\left(a(x)y'' + b(x)y'\right) = \left(r(x)y'\right)'$$
$$= ry'' + r'y'$$

Express μ, r through a, b

$$\mu a = r$$
$$\mu b = r'$$

$$(\mu a)' = \mu b$$
$$\mu a' + \mu' a = \mu b$$
$$\frac{\mu'}{\mu} = \frac{b-a'}{a}$$

$$(\ln \mu)'$$ logarithmic derivat.

$$\ln \mu = \int \frac{b-a'}{a}dx, \quad \boxed{\mu = \exp\left(\int\left(\frac{b-a'}{a}\right)dx\right)} \quad \mu \neq 0$$

$$r = a\mu.$$

[It's easier to repeat this computation each time]

Important: Verify the sign of r, p after the multiplication on $\mu(x)$.

Examples.

1. Cauchy-Euler eq.

$$x^2y'' + xy' - \alpha^2 y = 0$$

$$a = x^2, \quad b = x$$

$$\frac{b-a'}{a} = \frac{x-2x}{x^2} = -\frac{1}{x}$$

$$\mu(x) = \exp\left(-\int \frac{dx}{x}\right) = \exp(-\ln x) = \frac{1}{x}$$

$$\Downarrow$$

$$xy'' + y' - \alpha^2 \frac{y}{x} = 0$$

$$(xy')' - \alpha^2 \frac{y}{x} = 0$$

2. Bessel eq.

$$xy'' + y' + \alpha xy = 0$$

$$(xy')' + \alpha xy = 0$$

$$r = x, \quad q = 0, \quad p = x, \quad \lambda = \alpha$$

3. $x^2y'' + 3xy' + \lambda y = 0$

Guess: $\mu(x) = x \Rightarrow x^3y'' + 3x^2y' + \lambda xy = 0$

$$(x^3y')' + \lambda xy = 0.$$

$$r = x^3, \quad q = 0, \quad p = x.$$

$$\frac{b-a'}{a} = \frac{3x-2x}{x^2} = \frac{1}{x}$$

$$\mu(x) = \exp\left(\int \frac{dx}{x}\right) = e^{(\ln x)} = x.$$

Regular Sturm-Liouvill Problem

$$a_1 y(a) + B_1 y'(a) = 0$$
$$a_2 y(b) + B_2 y'(b) = 0 \qquad (2)$$

Regularity: (a_1, B_1), (a_2, B_2) are not ~~zeros~~ zero both

(other way there is no conditions!).

Theorem (St.-L.). (T. 12.5.1)

(i) There is an infinite number of eigen values

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots \to \infty \quad (n \to \infty)$$

(ii) For each j there is an unique eigen function $y_j(x)$ (up to multiplicative constant).

(iii) Eigen functions $y_j(x)$ for different λ_j are linear independent

(iv) They are orthogonal with the weight $p(x)$:

$$\int_a^b y_i(x)\, y_j(x)\, p(x)\, dx = 0, \quad i \neq j$$