2. (65 points) a) Give Euler's formula.

b) Expand the function $f(x) = e^{-x}$ in the complex Fourier series on $[-1, 1]$.

c) Describe the sum of this series for all x.

d) Transform this series in the real Fourier series using the formulas connecting coefficients of real and complex series.

a) $e^{ix} = \cos x + i\sin x$

b) $c_n = \frac{1}{2}\int_{-1}^{1} \exp(-x - in\pi x)\,dx = \frac{1}{2}\int_{-1}^{1} \exp(-x(1+in\pi))\,dx$

$= -\frac{1}{2}\frac{1}{1+in\pi}\exp(-x(1+in\pi))\Big|_{-1}^{1}$

$= -\frac{1}{2}\frac{1}{1+in\pi}\left[\exp(-1-in\pi) - \exp(1+in\pi)\right]$

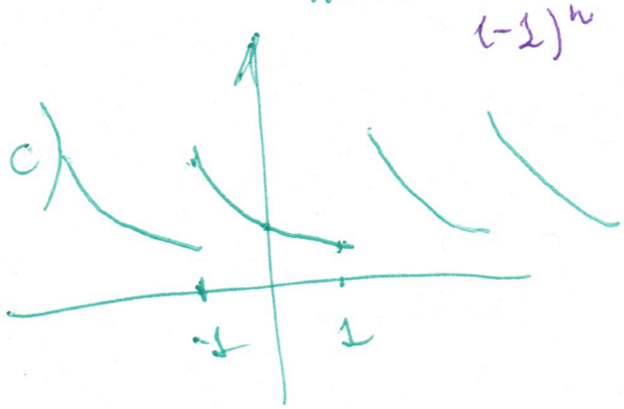
$e^{in\pi} = (-1)^n$

$= \frac{(-1)^n}{1+in\pi}\sinh(1).$

$$e^{-x} = \sinh(1)\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\underbrace{1+in\pi}}\exp(in\pi x), \quad -1<x<1.$$

$(-1)^n \frac{1-in\pi}{1+n^2\pi^2}$

c)



d) $a_n = c_n + c_{-n} = (-1)^n\sinh(1)\left(\frac{1}{1+in\pi} + \frac{1}{1-in\pi}\right) =$

$= \frac{2\sinh(1)}{1+n^2\pi^2}$, $a_0 = 2\sinh(1)$

$$b_n = i\left(c_n - c_{-n}\right) = 2\sinh(1)\,\frac{(-1)^{n+1}\, i n\pi}{1+n^2\pi^2}$$

$$e^{-x} = 2\sinh(1)\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2}\left(\cos(n\pi x) + n\pi \sin(n\pi x)\right)\right)$$

$-1 < x < 1,$

$$c_n = \frac{a_n - i b_n}{2}, \quad c_{-n} = \frac{a_n + i b_n}{2}, \quad n \geq 0.$$

Lecture 16.

General view on Fourier Series. Sect. 12.5

Eigen functions and eigen values on the line.

y_λ is an eigen function iff
(i) $y_\lambda(x) \not\equiv 0$
(ii) $y_\lambda'' + \lambda y = 0$.

(1) $\boxed{y'' + \lambda y = 0}$, $y(x)$, $-\infty < x < \infty$

General solution $y_\lambda(x)$

1) $\lambda \geq 0$, $\lambda = a^2$, $a > 0$ $y_\lambda(x)$

$y_a = c_1 \cos ax + c_2 \sin ax$.

2) $\lambda < 0$, $\lambda = -a^2$, $a > 0$

$y_a(x) = c_1 e^{ax} + C_2 e^{-ax}$

$= d_1 \cosh(ax) + d_2 \sinh(ax)$

$\cosh(ax)' = a \sinh(ax)$
$\sinh(ax)' = a \cosh(ax)$

3) $\lambda = 0$, $y'' = 0$

$y_0(x) = c_1 x + c_0$.

Eigen functions for boundary problems on segments.

Examples. 1. $y'' + \lambda y = 0$, $[0, \pi]$

$y(0) = y(\pi) = 0$ (2 conditions since the equation of 2nd order).

homogeneous cond

The selection of eigen functions, satisfying to the boundary conditions.

a) $\lambda > 0$, $\lambda = a^2$ $a = \sqrt{\lambda}$

$y_a(x) = c_1 \cos(ax) + c_2 \sin(ax)$

$y_a(0) = 0 \Rightarrow c_1 = 0 \Rightarrow y_a(x) = c \sin(ax)$

$y_a(\pi) = 0 \Rightarrow c \neq 0$ ($y_a \equiv 0$ is not an eigen function)

$\sin(a\pi) = 0 \Rightarrow a\pi = n\pi \Rightarrow$

$\Rightarrow a = n$, $n = 1, 2, \dots \Rightarrow$

$y_n(x) = \sin(nx)$, $n = 1, 2, \dots$, are the eigen functions with the eigen values $\lambda_n = n^2$.

b) $\lambda < 0$, $\lambda = -a^2$ $a = \sqrt{-\lambda}$

$y_a(x) = c_1 e^{at} + c_2 e^{-at}$

$y_a(0)=0 \Rightarrow c_1+c_2=0 \Rightarrow y_a(x)=c\,(e^{ax}-e^{-ax}),\ c\neq 0$

$y_a(\pi)=0 \Rightarrow e^{a\pi}-e^{-a\pi}=0 \Rightarrow e^{a\pi}=e^{-a\pi} \Rightarrow e^{2a\pi}=1 \Rightarrow$

$\Rightarrow a=0 \Rightarrow c=0$ Contradiction

No eigen functions with $\lambda<0$

c) $\lambda=0 \Rightarrow y_0(x)=c_1 x+c_0$

$y_0(0)=0 \Rightarrow c_0=0,\ y_0(x)=c_1 x$

$y_0(\pi)=0 \Rightarrow c_1\pi=0 \Rightarrow c_1=0,\ y_0\equiv 0$ Contradict.

No eigen functions with $\lambda=0$.

Only $y_n(x)=\sin nx$ are e. functions.

$n=1,2,3,\dots$

$\{\sin nx\}$ are base functions

for sine F. series on $[0,\pi]$.

Observation: $\{y_n(x)\}$ is the base of the Sine Fourier series

2. $y'' + \lambda y = 0$, $[0, p]$ (2)
$y'(0) = y'(p) = 0$.

Selection of e. functions

a) $\lambda > 0$, $\lambda = a^2$, $y_a(x) = C_1 \cos ax + C_2 \sin ax$

$y_a'(x) = -C_1 a \sin ax + C_2 a \cos ax$

$y_a'(0) = 0 \Rightarrow C_2 = 0$, $y_a(x) = C \cos ax$

$y_a'(p) = 0 \Rightarrow -C a \sin ap = 0$, $C \neq 0, a \neq 0$

$\Rightarrow \sin ap = 0 \Rightarrow ap = n\pi \Rightarrow a = \frac{n\pi}{p}$.

$y_n(x) = \cos\left(\frac{n\pi}{p}x\right)$, $n = 1, 2, \ldots$ are eigen functions with the eigen values $\lambda_n = \frac{n^2\pi^2}{p^2}$.

b) $\lambda < 0$, $\lambda = -a^2$

let us use hyperbolic functions (rather than exponential ones).

$y_a(x) = c_1 \cosh ax + c_2 \sinh ax$

$y_a'(x) = a c_1 \sinh ax + a c_2 \cosh ax$

$y_a'(0) = 0 \Rightarrow c_2 = 0 \Rightarrow y_a(x) = c\cosh ax$, $y_a'(x) = ac\sinh$
$a \neq 0, c \neq$

$y_a'(p) = 0 \Rightarrow \sinh(ap) = 0 \Rightarrow a = 0$ contradiction

$\sinh x = 0 \Leftrightarrow e^x = e^{-x} \Leftrightarrow e^{2x} = 1 \Leftrightarrow x = 0.$

No eigen functions with e.value $\lambda < 0$,

3) $\lambda = 0$ $\quad y_0(x) = c_1 x + c_0$

$y_0'(0) = 0 \Rightarrow c_1 = 0$, $y_0(x) = c$

So $y_0(x) = 1$ is an eigen function with $\lambda = 0$.

Unify cases 1) and 3):

$y_n(x) = \cos\left(\frac{n\pi}{p}x\right)$, $n = 0, 1, 2, \ldots$ are eigen functions for the problem (2) with the eigenvalues $\lambda_n = \frac{n^2\pi^2}{p^2}$. It correspond to the Cosine Fourier series.

Conclusion. Different boundary problems for the equation $y'' + \lambda y = 0$ correspond to different Fourier series:

Systems of eigen functions are always orthogonal and complete.