Lecture 1b

13.10

Sine F. series for $f(x) = \cos x$ on $[0,\pi]$

Cosine F. series $\cos x = \cos x$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) + \sin((n-1)x) \, dx$$

$$= \frac{1}{\pi}\left(-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right)\Big|_0^{\pi}$$

$$-\frac{1}{n+1}\left((-1)^{n+1} - 1\right)$$

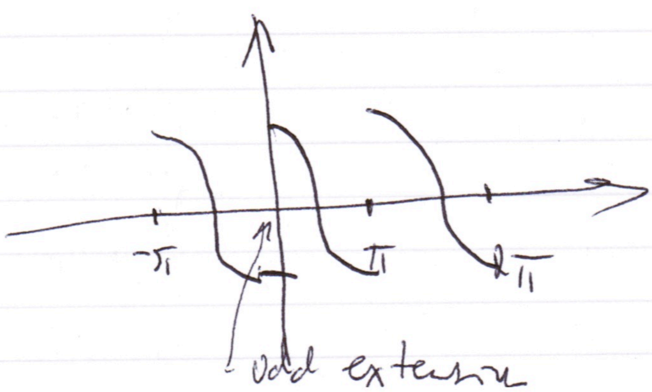
$$= \begin{cases} \frac{2}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) & n \text{ even} \\ 0 & \text{odd} \end{cases} \quad n \geq 2$$

$$= \begin{cases} \frac{4}{\pi}\frac{n}{n^2-1}, & n \text{ even} \\ 0 & n > 1 \text{ odd} \end{cases}$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx = 0$$

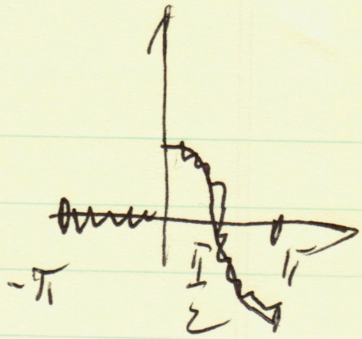
$$\cos x = \frac{4}{\pi}\sum_{n \text{ even}} \frac{n}{n^2-1} \sin nx \, .$$

$0 < x < \pi$



π-periodic

$$f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ 0, & -\pi < x < 0 \end{cases}$$



$$f(x) = \frac{\cos x}{2} + \frac{2}{\pi} \sum_{n \text{ even}} \frac{n}{n^2-1} \sin nx$$

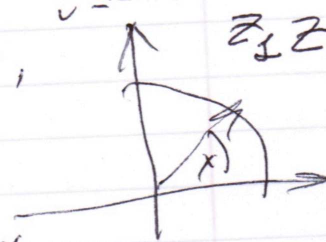
Lecture 15. (Section 12.4).

Complex Fourier Series.

$z = x + iy$

$i^2 = -1$

$z_1 z_2$



Euler's formula: $\sqrt{-1} = \pm i$,

$e^{x_1+x_2} = e^{x_1} e^{x_2}$

$$e^{ix} = \cos x + i \sin x, \quad x \in \mathbb{R}$$

$e^{i(x_1+x_2)} = e^{ix_1} e^{ix_2}$

[compatible with series]

$\cos(x_1+x_2) + i\sin(x_1+x_2) =$
$= (\cos x_1 \cos x_2 - \sin x_1 \sin$ ...
$+ i(\sin x_1 \cos x_2 +$ ...
$\cos x_1 \sin$ ...

$e^{2\pi i} = 1$, $e^{\pi i} = -1$, $e^{n\pi i} = (-1)^n$

$e^{\frac{\pi}{2} i} = i$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$e^{i(z_1+z_2)} = e^{iz_1} e^{iz_2}$ $\quad \underline{e^z = e^x} (\cos y + i \sin y)$, $\cos x = \cosh ix$, $\sin x = \sinh i$ ...

$[-p, p]$. $\quad \exp\left(i \frac{n\pi x}{p}\right)$ is a complet orthogonal system

$\cos\left(\frac{n\pi x}{p}\right) + i\sin\left(\frac{p n p x}{\pi}\right)$

$-\infty < n < \infty$ $\quad$ $2p$ - periodic

Complex Fourier series

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \exp\left(i \frac{n\pi x}{p}\right)$$

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) \exp\left(-i \frac{n\pi x}{p}\right) dx.$$

Example.

$f(x) = \exp(2x)$, $[-\pi, \pi]$. $\exp(inx)$ $-\infty < n < \infty$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(x(2-in))\,dx =$$

$$= \frac{1}{2\pi}\,\frac{1}{2-in}\exp(x(2-in))\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(2-in)}\left(\exp(2\pi - i\pi n) - \exp(-2\pi + i\pi n)\right)$$

$e^{i\pi n} = e^{-i\pi n} = (-1)^n$

$$= \frac{(-1)^n}{2\pi(2-in)}\left(\exp 2\pi - \exp(-2\pi)\right)$$

$$c_n = \boxed{\frac{(-1)^n}{\pi(2-in)} \sinh 2\pi}$$

$$\exp(2x) = \frac{\sinh 2\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2-in}e^{inx}$$

$-\pi < x < \pi$

$$c_0 = \frac{\sinh(2\pi)}{2\pi}$$

$$\frac{1}{2-in} = \frac{2+in}{4+n^2}$$

$$(x+iy)(x-iy) = x^2+y^2$$

$$\frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

Properties.

f - even $\sim$ $c_n = c_{-n}$

f - odd $\sim$ $c_n = -c_{-n}$ (including $c_0 = 0$)

$$\exp\left(i \frac{n\pi x}{p}\right) = \cos\left(\frac{n\pi x}{p}\right) + i \sin\left(\frac{n\pi x}{p}\right)$$

$$\exp\left(-i \frac{n\pi x}{p}\right) = \cos\left(\frac{n\pi x}{p}\right) - i \sin\left(\frac{n\pi x}{p}\right)$$

$$a_n = c_n + c_{-n}, \quad a_0 = 2c_0$$

$$b_n = i(c_n - c_{-n})$$

$$c_n = \frac{a_n - ib_n}{2} \qquad c_0 = \frac{a_0}{2}$$

$$c_{-n} = \frac{a_n + ib_n}{2} \qquad c_{-n} = \overline{c_n}$$

(for real f)

$z = x + iy$, $\bar{z} = x - iy$

We can transform complex Fourier series at real ones and back!

$f(x) = \exp(2x)$, $[-\pi, \pi]$ $\quad a_n = c_n + c_{-n}$

$$a_n = \frac{\sinh(2\pi)}{\pi}(-1)^n\left(\frac{1}{2-in} + \frac{1}{2+in}\right) =$$

$$\underbrace{(-1)^n = (-1)^{-n}}\; = \boxed{\frac{4\sinh 2\pi}{\pi}\,\frac{(-1)^n}{n^2+4}}$$

$b_n = i(c_n - c_{-n})$

$$b_n = \frac{\sinh(2\pi)}{\pi}\frac{(-1)^n}{i}\left(\frac{1}{2-in} - \frac{1}{2+in}\right) =$$

$$= \boxed{\frac{\sinh(2\pi)}{\pi}\,(-1)^n\,\frac{2n}{4+n^2}}$$

$$\exp(2x) = \frac{\sinh(2\pi)}{\pi}\left(\frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+4}\cos nx\right.$$

$-\pi < x < \pi$

$$\left. + 2\sum_{m=1}^{\infty}\frac{(-1)^m m}{m^2}\sin mx\right)$$