b) Prove that the given system of functions

$$1, \cos 2x, \cos 4x, \cos 6x, \ldots, \cos 2nx, \ldots$$

is orthogonal on $[0, \pi/2]$.

c) Find the norms of each of these functions.

d) Are functions $\cos(2\pi kx)$ orthogonal on $[0, \pi/2]$? Give a detailed explanation.

a) $f \perp g \Leftrightarrow (f,g) = \int_a^b f(x)\, g(x)\, dx = 0.$

b) $\displaystyle\int_0^{\pi/2} \cos(2kx)\cos(2\ell x)\,dx = \frac{1}{2}\int_0^{\pi} \cos(ky)\cos(\ell y)\,dy$

$k \neq \ell$

$\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$.

$\displaystyle = \frac{1}{4}\left(\int_0^{\pi} \cos((k+\ell)y) + \cos((k-\ell)y)\right)dy$

$\displaystyle = \frac{1}{4}\left(\frac{\sin(k+\ell)y}{k+\ell} + \frac{\sin(k-\ell)y}{k-\ell}\right)\Bigg|_0^{\pi} = 0$, since $\sin n\pi = 0$

c) $\displaystyle \|f\|^2 = \frac{1}{2}\int_0^{\pi} \cos^2 ky\,dy = \frac{1}{4}\int_0^{\pi} (\cos(2ky) + 1)\,dy = \begin{cases} \pi/4, & k \neq 0 \\ \pi/2, & k = 0. \end{cases}$

d) $\displaystyle (1, \cos 2\pi x) = \int_0^{\pi/2} \cos(\cancel{\cos}\, 2\pi x)\,dx = \frac{1}{2\pi}\sin(2\pi x)\Big|_0^{\pi/2} \neq 0. = \frac{1}{2\pi}\sin(\pi^2) \neq 0$

$\displaystyle (\cos 2\pi n x, \cos 2\pi m x) = \int_0^{\pi/2} \cos 2\pi nx \cos 2\pi mx\,dx =$
$\displaystyle = \frac{1}{2}\int_0^{\pi} (\cos 2\pi(m+n)x + \cos 2\pi(m-n)) = \frac{1}{4\pi}\left(\frac{1}{m+n}\sin(2\pi(m+n))x\right.$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. (100 points) a) Expand the function on $(-\pi, \pi)$:

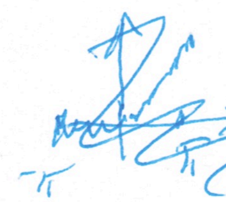
$$f(x) = 0, -\pi < x < 0, f(x) = \pi - x, 0 < x < \pi$$

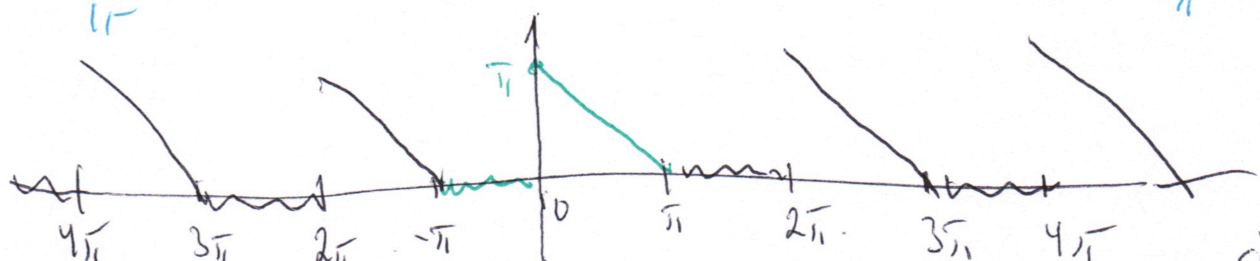
in the Fourier series.

b)Specify the numerical series which are the results of evaluations for $x = 0, \pi/2$.

c)Describe the sum of these series for all $-\infty < x < \infty$. Give explanations and sketch the graphs.

d) Consider the Fourier series which keeps only the terms with the sinuses of our series. What is the sum of this series? Sketch the graph of this function.





c)

$$a_0 = \frac{1}{\pi}\int_0^\pi (\pi - x)\,dx = \boxed{\frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_0^\pi (\pi - x)\cos nx\,dx = \frac{1}{\pi}\left(\pi\int_0^\pi \cos nx\,dx - \int_0^\pi x\cos nx\,dx\right)$$

$n \neq 0$

$$= \frac{1}{\pi}\left(\underbrace{\frac{\pi}{n}\sin nx\Big|_0^\pi}_{=0} - \underbrace{\frac{x}{n}\sin nx\Big|_0^\pi}_{=0} + \frac{1}{n}\int_0^\pi \sin nx\,dx\right)$$

$$= -\frac{1}{\pi n^2}\cos nx\Big|_0^\pi = \boxed{\frac{1}{\pi n^2}\left(1-(-1)^n\right)},$$

$$b_n = \frac{1}{\pi}\left(\pi\int_0^\pi \sin nx\,dx - \int_0^\pi x\sin nx\,dx\right) = \frac{1}{\pi}\left(-\frac{\pi}{n}\cos nx\Big|_0^\pi + \right.$$

$$\left. + \frac{x}{n}\cos nx\Big|_0^\pi - \frac{1}{n}\underbrace{\int_0^\pi \cos nx\,dx}_{=0}\right) = \frac{1}{\pi}\left(\frac{\pi(1-(-1)^n)}{n} + \frac{\pi}{n}(-1)^n\right)$$

$$= \boxed{\frac{1}{n}}$$

$$f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2}\cos nx$$
$-\pi < x < \pi$

$$+ \sum_{m=1}^{\infty} \frac{1}{m}\sin mx .$$

$a_n = 0$ for even n

$\cos\frac{n\pi}{2}$, n odd

$\sin mx = \begin{cases} 0, m - e \\ (-1)^k, \\ m = 2 \end{cases}$

---

b) $x = 0$

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} \quad \Bigg| \quad \frac{\pi^2}{8} = \sum_{\text{odd } n} \frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \dots$$

↑ "jump"

$x = \frac{\pi}{2}$
$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} \quad \Bigg| \quad \frac{\pi}{4} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - - - -$$

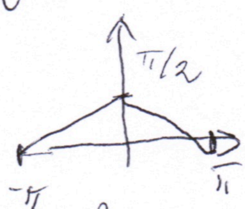
n even $-$ $a_n = 0$

n odd, $\cos n\frac{\pi}{2} = 0.$

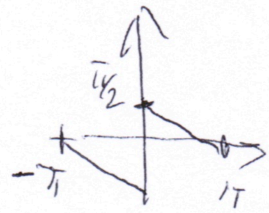
$$\sin\frac{\pi}{2}(2k+1) = (-1)^k$$

m ~~odd~~ even $\sin m\frac{\pi}{2} = 0$

d) $f = f_{even} + f_{odd}$



feven



$$f_e = \frac{\pi - |x|}{2}$$

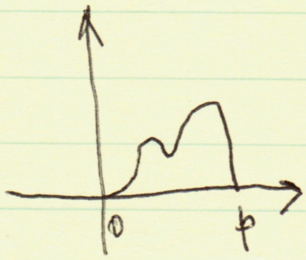
$$f_{odd} = \cancel{\frac{(\pi - |x|)}{2}\,\text{sgn}}$$
$$= \begin{cases} \frac{\pi - x}{2}, & x > 0 \\ \frac{-\pi - x}{2}, & x < 0 \end{cases}$$
$$= \left(\frac{\pi - |x|}{2}\right)\text{sgn}\, x$$

Sum of sinuses $-$ f_{odd}

$$f_{even}(x) = \frac{1}{2}\left(f(x) + f(-x)\right)$$

$$f_{odd}(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

Let $f(x)$ be a function on $[0,p]$



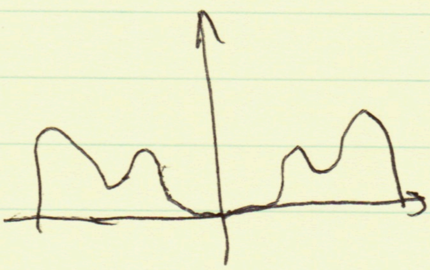
"Can be extended at Sine and Cosine F. series

1) Even extension on $[-p,p]$

$$f_1(x) = \begin{cases} f_1(x) = f(x), & x \in [0,p] \\ f_1(x) = f(-x), & x \in [-p,0] \end{cases}$$

~~Cosine F series~~ ~~$\Leftrightarrow$~~

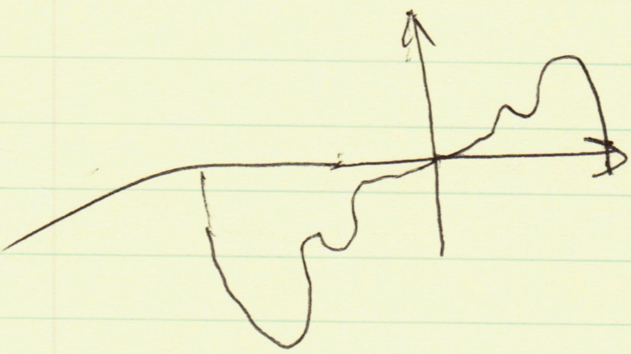
Symmetry relative y-axis



Cosine F. series for $f(x)$ $\Leftrightarrow$ Fourier series for $f_1(x)$

2) Odd extension

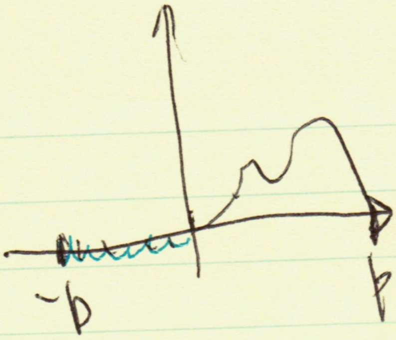
$$f_2(x) = \begin{cases} f_2(x) = f(x), & x \in [0,p] \\ f_2(x) = -f(-x), & x \in [-p,0] \end{cases}$$



Symmetry relative of the origin.

Sine F series for $f(x)$ $\Leftrightarrow$ F. series for $f_2(x)$.

3) $[-p,p]$



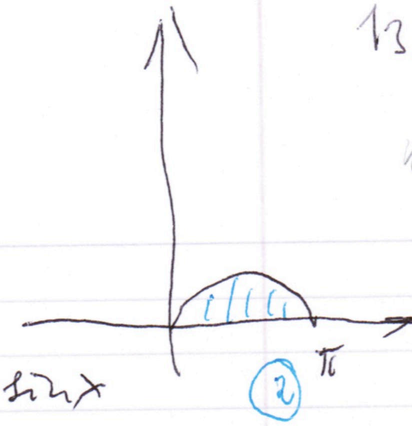
$$f_3(x) = \begin{cases} f(x), & x \in [0,p] \\ 0, & x \in [-p,0] \end{cases}$$

$$f_3(x) = \frac{1}{2}\left(f_1(x) + f_2(x)\right).$$

F. series for f_3 is the half-sum of Cosine and Sine F. series

13.

$f(x) = \sin x, \quad 0<x<\pi$



Sine series is trivial. $\sin x \equiv \sin x$

f_2 (odd extension)

Fine Cosine F. series

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx =$$

$n \neq 0$

$$= \frac{1}{\pi}\int_0^{\pi} (\sin (n+1)x - \sin (n-1)x)\, dx$$

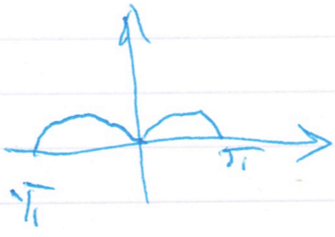
$n>1$
$$= \frac{1}{\pi}\left(-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right)\Bigg|_0^{\pi}$$

$\cos m\pi = (-1)^m$

$$= \begin{cases} \frac{2}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = -\frac{4}{\pi}\frac{1}{n^2-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$a_1 = 0$

Even extension $f_1(x)$



$f_1(x) = |\sin x|$.

13.

Other examples:

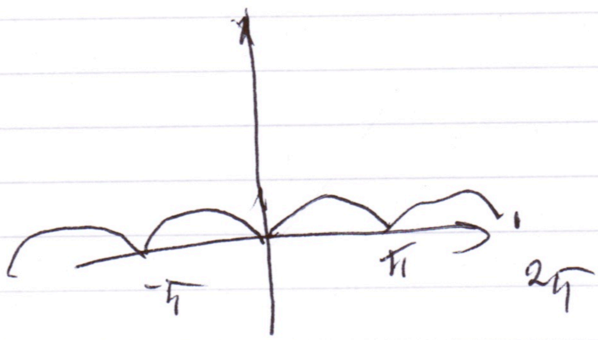
$$\sin x = \frac{2}{\pi}\left(1 - 2\sum_{\substack{n \text{ even} \\ n>0}} \frac{\cos nx}{n^2-1}\right) =$$

$0<x<\pi$

$$= \frac{2}{\pi}\left(1 + \sum_{n>0 \text{ even}} \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\cos nx\right).$$

$x=0$ $\quad \frac{1}{2} = \sum_{n>0 \text{ even}} \frac{1}{n^2-1}$

$$0 = \left(1 + \frac{1}{3} - 1 + \frac{1}{5} - \frac{1}{3} \cdots\right)$$



$|\sin x|$

$-\infty < x < \infty$

P2. 9. (12.2)

$$f(x) = \begin{cases} 0, & \cancel{0<x<\pi} \quad -\pi < x < 0 \\ \sin x, & \cancel{0<} \quad 0 < x < \pi \end{cases}$$

$$f(x) = \frac{1}{2}\left(\sin x + |\sin x|\right).$$

$$= \frac{\sin x}{2} + \frac{1}{\pi}\left(1 - \sum_{\text{even } n} \frac{\cos nx}{n^2-1}\right)$$

$\frac{1}{2}\sin x$

$1/2 \sin x$