Cosine and Sine Fourier Series

Sect. 1

- $\left\{\cos\frac{n\pi x}{p}, \sin \frac{m\pi x}{p}\right\}$ is complet on $[-p,p]$

Each of

$$\left\{\cos\frac{n\pi x}{p}\right\} \text{ and } \left\{\sin\frac{m\pi x}{p}\right\}$$

are complet on $[0,p]$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p} \quad \text{[cosine seri}$$

$$= \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{2} \quad \text{[sine series]}$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx$$

$$b_m = \frac{2}{p}\int_0^p f(x)\sin\frac{m\pi x}{p}\,dx$$

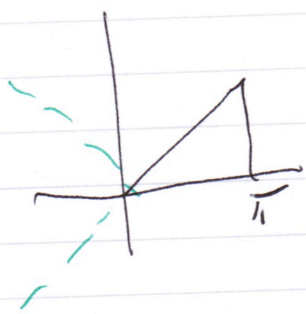
Fourier S. on $[-p,p]$ for even functions $\Longleftrightarrow$ Cosine F. series on $[0,p]$.

-"- for odd functions $\Longleftrightarrow$ Sine F series on $[0,p$

The same function $f(x)$ on $[0,p]$ can be expand at both Cosine and Sine Fourier series.

Examples.

$p=\pi \qquad f(x)=x$



Even exten $|x|$

Odd one x

Cosine F. series

$$x = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cos nx$$

$0<x<$

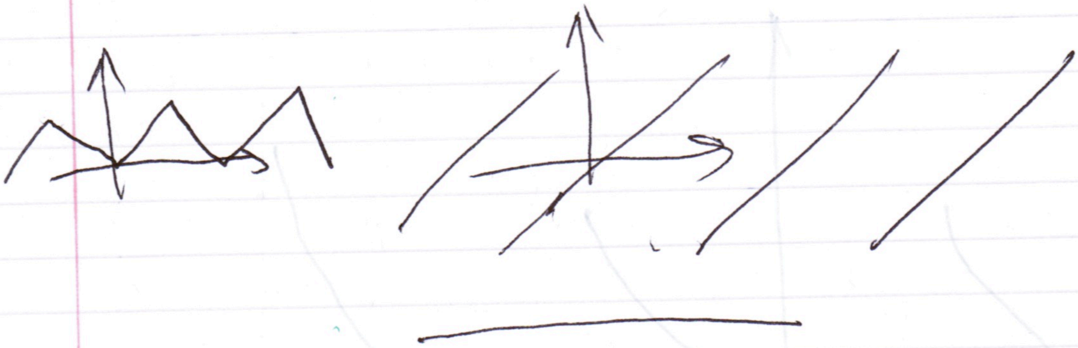
Sine F. series

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$$

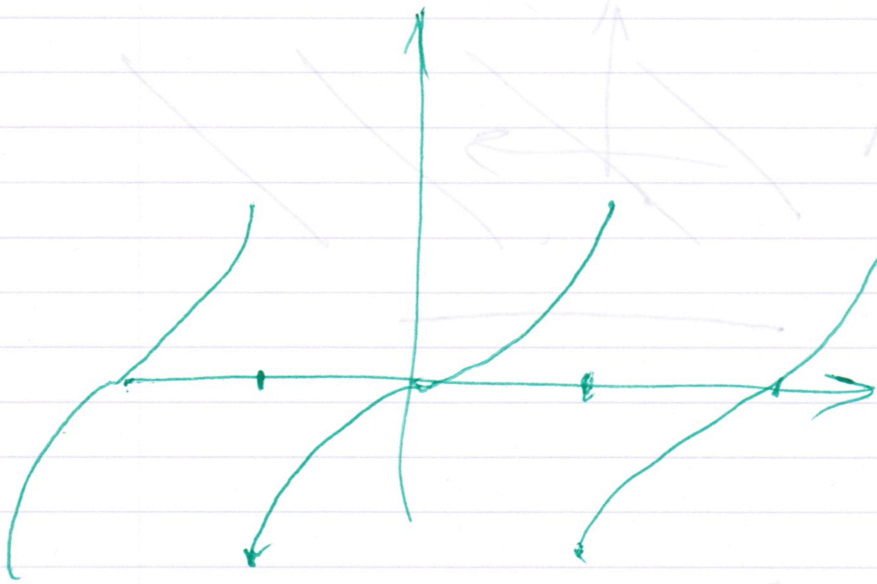
~~13.3.~~ 4.

~~13.5~~

Extensions of series on $(-\infty, \infty)$ are different:



$y = x^2$



13.6.

$$f_1(x) = \cancel{f(x)}\, x^2, \quad -\pi < x < \pi \quad p = \pi$$

4

$$\underset{n\neq 0}{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \underbrace{\frac{1}{\pi}\, \frac{x^2 \sin nx}{n}\Bigg|_{-\pi}^{\pi}}_{=0}$$

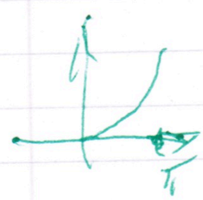
$$- \frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin n\cancel{x}$$

$$= \frac{2(-1)^{n+1}}{n}\pi \quad \text{(old computation)}$$

$$a_n = \frac{4(-1)^n}{n^2}, \quad n \neq 0$$

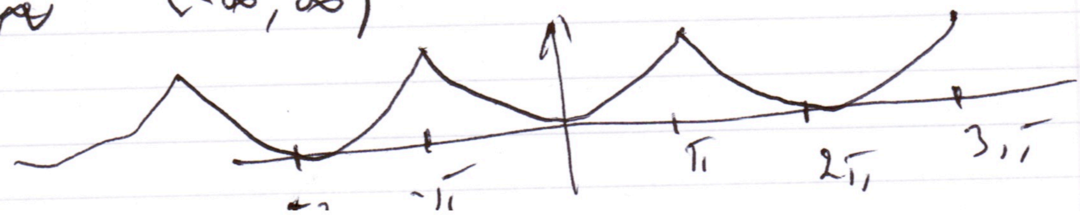
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^2}{3}$$

$$\boxed{x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \quad -\pi < x < \pi}$$



Corollary. It's the Cosine Fourier series on $[0, \pi]$

On $(-\infty, \infty)$



13.7

7

$f(x) = x^2$, $[0,\pi]$. What is Cosine F. series?

Find Sine F. series

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx = -\frac{2x^2\cos nx}{\pi n}\Big|_0^{\pi}$$

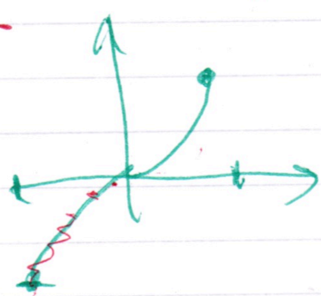
$$+ \frac{4}{\pi n}\int_0^{\pi} x\cos nx \, dx \quad \leftarrow \text{already computed}$$

$$= \frac{2\pi(-1)^{n+1}}{n} + \frac{4((-1)^n - 1)}{\pi n^3}$$

$$x^2 = \sum_{n=1}^{\infty}\left(\frac{2\pi(-1)^{n+1}}{n} + \frac{4((-1)^n-1)}{\pi n^3}\right)\sin nx.$$

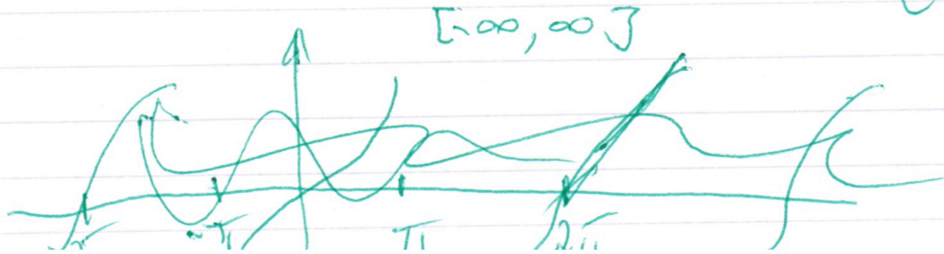
$0 < x < \pi$

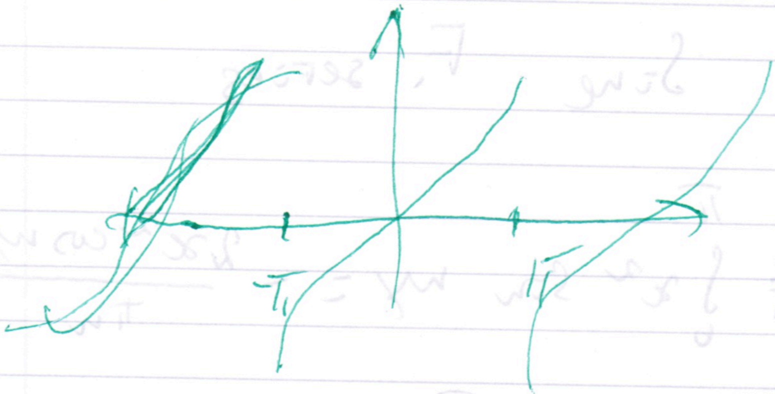
Odd extension on $[-\pi,\pi]$



$$\text{sgn}\, x \cdot x^2 = \begin{cases} x^2 & x > \\ -x^2 & x < \end{cases}$$

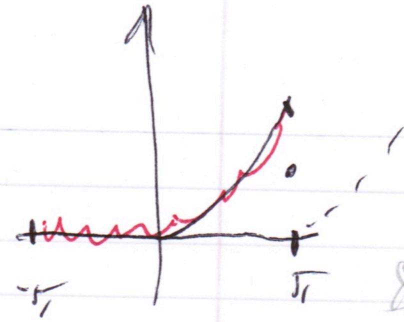
$[-\infty,\infty]$




-π
π

13,8

$$f(x) = \begin{cases} x^2, & 0 < x < \pi \\ 0, & -\pi < x < 0 \end{cases}$$



[Expend at F. series on $[-\pi, \pi]$]

$$a_n = \frac{2(-1)^n}{n^2}, \; n \neq 0, \qquad a_0 = \frac{\pi^2}{3}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx = -\left.\frac{x^2 \cos nx}{\pi n}\right|_0^{\pi}$$

$$+ \frac{2}{\pi n}\int_0^{\pi} x \cos nx \, dx =$$

$$= \frac{\pi(-1)^{n+1}}{n} + \frac{2((-1)^n - 1)}{\pi n^3}.$$

$$\boxed{\begin{aligned} f(x) &= \frac{1}{2}\left(f_1(x) + f_2(x)\right) \\ &= f_{even} + f_{odd} \end{aligned}}$$

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$+ \sum_{m=1}^{\infty}\left(\frac{\pi(-1)^{m+1}}{m} + \frac{2((-1)^m - 1)}{\pi m^3}\right) \sin mx$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$\sin (2k+1)\frac{\pi}{2} = (-1)^k$

$\cos (2k+1)\frac{\pi}{2} = 0$