MATH 421. ADVANCED CALCULUS FOR ENGINEERING. FALL 2014. MIDTERM 1

1. (45 points)

Find given Laplace transforms

$\mathcal{L}\{te^{-t}\sinh(3t)\}$;

$\mathcal{L}\{\mathcal{U}(t-2)t^2e^{-t}\}$;

$\mathcal{L}\{e^{2t}\delta(t-3)\}$.

① $\mathcal{L}(\sinh(3t)) = \dfrac{3}{s^2-9}$

$\mathcal{L}(t\sinh(3t)) = -3\left(\dfrac{1}{s^2-9}\right)' = -3\left(\dfrac{-2s}{(s^2-9)^2}\right) = \underline{\dfrac{6s}{(s^2-9)^2}}$

$\mathcal{L}(te^{-t}\sinh(3t)) = \dfrac{6(s+\cancel{6}1)}{(s^2+2s-8)^2}$;

② $g(t) = t^2e^{-t}$; $g(t+2) = e^{-2}e^{-t}(t+2)^2 = e^{-2}e^{-t}(t^2+4t+4)$

$a=2$

$$\mathcal{L}(\mathcal{U}(t-2)t^2e^{-t}) = \underline{e^{-2s-2}\left(\frac{2}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4}{s+1}\right)}.$$

③ $\cancel{\delta}$ $\mathcal{L}(\delta(t-3)) = e^{-3s} \Rightarrow \mathcal{L}(\cancel{\delta}e^{2t}\delta(t-3)) = \underline{e^{-3(s-2)}}$.

2nd sol.: $e^{2t}\delta(t-3) = e^6\delta(t-3) \Rightarrow \mathcal{L}(e^{2t}\delta(t-3)) = e^6e^{-3s}$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. (45 points)
Find given inverse Laplace transforms

$$\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+9)(s^2-4)}\right) = \frac{3}{13}\sin 3t + \frac{2}{13}\sinh 2t.$$

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-6s+11}\right).$$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}(s^2-1)}{s^2-2s+2}\right).$$

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$$\frac{s^2}{(s^2+9)(s^2-4)} = \frac{1}{13}\left(\frac{9}{s^2+9} + \frac{4}{s^2-4}\right)$$

2

$$= \mathcal{L}^{-1}\left(\frac{e^{-3s}}{(s-3)^2+2}\right) = \frac{\sqrt{2}}{2}u(t-3)e^{3(t-3)}\sin\sqrt{2}(t-3).$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2+2}\right) = \frac{\sqrt{2}}{2}e^{3t}\sin\sqrt{2}t$$

3

$$\frac{s^2-1}{s^2-2s+2} = 1 + \frac{2s-3}{(s-1)^2+1} = 1 + \frac{2(s-1)-1}{(s-1)^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}(s^2-1)}{s^2-2s+2}\right) = \delta(t-1) + u(t-1)e^{t-1}\left(2\cos(t-1) - \sin(t-1)\right).$$

3. (40 points) Solve given initial-value problem using the Laplace transform
$y'' - 2y' + 5y = e^t t, y(0) = -1, y'(0) = -2;$

$$(s^2 - 2s + 5)Y = \frac{1}{(s-1)^2} - s$$

$$[(s-1)^2 + 4]Y = \frac{1}{(s-1)^2} - (s-1) - 1$$

$$Y = \frac{1}{(s-1)^2\underbrace{[(s-1)^2+4]}_{u}} - \frac{s-1}{[\quad]} - \frac{1}{[\;]}$$

$$u = \frac{1}{(s-1)^2}; \quad \frac{1}{u(u+4)}$$

$$y = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2} - \frac{1}{[(s-1)^2+4]}\right) - e^t\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

$$y = e^t\left(\frac{t}{4} - \frac{1}{8}\sin 2t - \cos 2t - \frac{\sin 2t}{2}\right)$$

$$= e^t\left(\frac{t}{4} - \frac{5}{8}\sin 2t - \cos 2t\right).$$

$$y(0) = -1$$

4. (35 points) Solve given integral equation
$y(t) = te^{-t} + \int_0^t y(\tau)\sin(2(t-\tau))d\tau.$

$$Y = \frac{1}{(s+1)^2} + \frac{2}{s^2+4} Y$$

$$\left(\frac{s^2+2}{s^2+4}\right) Y = \frac{1}{(s+1)^2}$$

$$Y = \frac{s^2+4}{(s+1)^2(s^2+2)} = \frac{as+B}{(s+1)^2} + \frac{Cs+D}{s^2+2}$$

$a + C = 0$

$B + 2C + D = 1$

$2a + C + 2D = 0 \quad | \; a + 2D = 0$

$2B + D = 4$

$a = \frac{4}{9}$

$B = \frac{19}{9}$

$C = -\frac{4}{9}$

$D = -\frac{2}{9}$

$$y = e^{-t}\left(at + (B-a)\right) + C\cos(\sqrt{2}t) + \frac{D}{\sqrt{2}}\sin(\sqrt{2}t)$$

$$= \frac{1}{9}\left(4e^{-t} + 15e^{-t}\cdot t - 4\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t)\right)$$

5. (35 points) Solve the initial-value problem for given system of differential equations using the Laplace transform

$$x' = 2x + y, y' = x + 2y, x(0) = 1, y(0) = -2.$$

$sX - 1 = 2X + Y$

$sY + 2 = X + 2Y$

$X(s-2) - Y = 1$

$-X + (s-2)Y = -2$

$\cancel{(s-2)^2}$ $X((s-2)^2 - 1) = s - 2 - 2$

$X = \frac{(s-2)-2}{(s-2)^2 - 1} =$ $\boxed{x(t) = e^{2t}(\cosh t - 2\sinh t) = \frac{1}{2}(-e^{3t} + 3e^{t})}$

$= \frac{s-4}{(s-3)(s-1)} = \frac{1}{2}\left(+\frac{3}{s-1} - \frac{1}{s-3}\right)$

$y(t) = e^{2t}(-2\cosh t + \sinh t)$

$= -\frac{1}{2}(e^{3t} + 3e^{t})$

$y(t) = x'(t) - 2x(t)$

$=$

1

Lecture 11. Sect. 12.2

① Why orthogonal systems are convenient?

Good formulas for coefficients of series.

$\{\varphi_1, \ldots, \varphi_n, \ldots\}$ is an orthogonal system ~~s~~ and

$$f = \sum_{n=1}^{\infty} a_n \varphi_n \qquad (*)$$

completness

"enough of φ_j"

Then

$$a_n = \frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)}. \qquad (**)$$

Just take the inner product with φ_n with $(*)$.

On the right only one term ~~s~~ survives!

$$(\varphi_n, f) = a_n (\varphi_n, \varphi_n).$$

In our case!

$$a_n = \frac{\int_a^b f(x) \varphi_n(x)\,dx}{\int_a^b |\varphi_n(x)|^2 dx}. \qquad (**\cancel{*})$$

We have on $[-\pi, \pi]$ the orthogonal system

$$\{ \sin nx,\ n = 1, \dots ;\ \cos mx,\ m = 0, 1, \dots \}.$$

We don't prove the completeness of this system!

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (**)$$

Fourier series (or trigonometric series)

Then $a_n = \dfrac{(f, \cos nx)}{\pi}$, $n = 0, 1, 2, \dots$,

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$b_m = \dfrac{(f, \sin mx)}{\pi}$, $m = 1, \dots$

$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx.$

Why we take $\frac{a_0}{2}$?

Trigonometric series or Fourier series

If we have a continues function $f(x)$, $x \in [-\pi, \pi]$

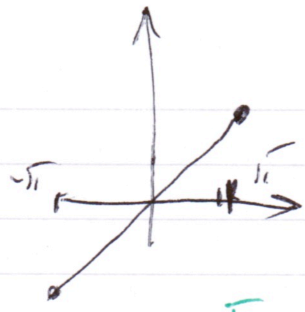
the we can compute a_n, b_m & consider the series $(**)$

It will pointwise convergent to $f(x)$ (without proof). [at each point]

$[-\pi,\pi]$

Examples.

1. $f(x) = x$



odd!

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \quad \Big| \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \, dx = 0 \quad \forall n$$

$n = 1, 2, \ldots$

$$= \frac{1}{\pi}\left(- \frac{x \cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\underbrace{\int_{-\pi}^{\pi} \cos nx \, dx}_{(1,\cos nx) = 0} \right)$$

$$= \frac{1}{\pi}\left(- \frac{\pi}{n}\big(\underbrace{\cos n\pi}_{(-1)^n} + \cos(-n\pi) \big) \right) = \frac{2(-1)^{n+1}}{n} = b_n$$

$$\boxed{x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}}$$

$-\pi < x < \pi$

$$= 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots \right)$$

Special points!

$x = 0 \qquad 0 = 0$

$x = \frac{\pi}{2} \qquad \frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$

n even $\qquad \sin k\pi = 0.$

$n = 2k$