Part 1 (of 3)

The Laplace Transform

Ch. 4, AEM (textbook)

Lecture 1. The definition and examples

Functional transforms (operators)

$$f(t) \Rightarrow \mathcal{A}f(s)$$

$\mathcal{A}f(s) = F(s)$

$\mathcal{A}$ - transform or operator

{functions transform at functions}

(compere with geom. transforms)

Why: transform differential equations at algebraic ones

Definition. Let $f(t)$ be a function for $0 \le t < \infty$. Then

$$F(s) = (\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

is called the Laplace transform of f.

Usually, (i) $0 < s < \infty$

(ii) f is - continuous (seldom piecewise)

- absolutely bounded

$\exists M$ such that
exists $|f(t)| < M$, $0 \le t \le \infty$

Under these conditions
$\mathcal{L}f$ is well defined
(sufficient conditions)

$\Leftarrow$ Calculus: Improper integrals.

$$\int_0^\infty f(t)\,dt = \lim_{T \to +\infty} \int_0^T f(t)\,dt$$

$T > 0$

Examples: Compute $\mathcal{L}f$

1. $f(t) \equiv 1$.

$$F(s) = \mathcal{L}f(s) = \lim_{T\to+\infty} \int_0^T e^{-st}\,dt =$$

$$= \lim_{T\to+\infty} \left\{ \frac{1}{s}(-e^{-st}) \right\}\Big|_0^T$$

$$= \lim_{T\to+\infty} \frac{1}{s}\left[-e^{-sT}+1\right] = \frac{1}{s}$$

Condition $s>0$ is essential!

2. $f(t)=t$

$$\mathcal{L}f = \int_0^\infty \underset{u}{t}\,\underset{v'}{e^{-st}}\,dt \underset{s>0}{=} \underbrace{-\frac{te^{-st}}{s}\Big|_0^\infty}_{!!\ =0} + \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{\text{Ex. 1}}$$

$$= \frac{1}{s^2}.$$

Compare the growth of t, e^{st} if $t\to+\infty$

$$\boxed{\text{Integration by parts: } \int uv'\,dt = uv - \int u'v\,dt} \Rightarrow \text{Calculus}$$

3 Generalization of Ex. 1, 2:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad n \geq 0, \; s > 0.$$

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4 $f(t) = e^{at}$, $a \leq 0 \Rightarrow$ decreases

$$\mathcal{L}f(s) = \int_0^\infty e^{at} e^{-st} \, dt = \int_0^\infty e^{-(s-a)t} \, dt = \frac{1}{s-a}$$

Essental condition $a \leq 0$ (Why?)

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5 $f(t) = \sin t$, $g(t) = \cos t$ then f is bounded $\mathcal{L}f, \mathcal{L}f$ are existed

Compute simulteneosly

$$A = \mathcal{L}f, \qquad B = \mathcal{L}g$$

$$A = \int_0^\infty \underset{v'}{\sin t} \; \underset{u}{e^{-st}} \, dt = -\cos t \, e^{-st} \Big|_0^\infty - s \int_0^\infty \cos t \, e^{-st} \, dt$$

Intgert. by parts

$$= 1 - sB$$

$$B = \int_0^\infty \cos t \, e^{-st} \, dt = \sin t \, e^{-st} \Big|_0^\infty + s \int_0^\infty \sin t \, e^{-st} \, dt$$

$$= sA$$

$$\left.\begin{array}{l} A = 1 - sB \\ B = sA \end{array}\right\} \Rightarrow A = 1 - s^2 A \Rightarrow A = \frac{1}{1+s^2} \Rightarrow B = \frac{s}{1+s^2}$$

Table 1

Let us to start a table

$f(t)$	$\mathcal{L}f(s)$
t^n, $n=0,1,\dots$	$\frac{n!}{s^{n+1}}$
e^{at}, $a<0$	$\frac{1}{s-a}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$, $s \geq \lvert k \rvert$
$\cosh kt$	$\frac{s}{s^2-k^2}$, $s \geq \lvert k \rvert$