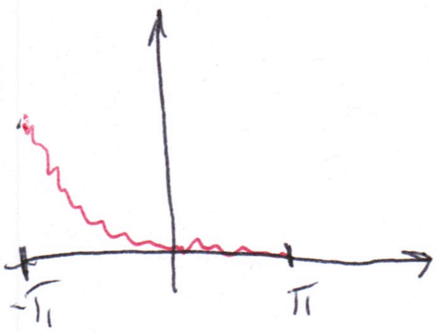
MATH 421.03. ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. MIDTERM 2

1. (60 points) 1) Find the Fourier series on the interval $-\pi < x < \pi$ of the function $f(x) = x^2$ for $-\pi < x < 0$ and $f(x) = 0$ for $-0 < x < \pi$.

2) Specify the numerical series corresponding to the points $x = 0$ and $x = \pi$.

3) Describe the function which this series represents on the whole line $-\infty < x < \infty$.

4) Let us keep in the Fourier series only terms with Cosines. Which function will represent this new series? Give complete explanations.



$$p = \pi. \qquad a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \boxed{\frac{\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x^2 \cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx =$$

$$= \frac{1}{\pi n} x^2 \sin(xn)\Big|_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi} x\sin(nx)\,dx$$

$$\overset{\parallel}{0}$$

$$= \frac{2}{\pi n^2} x\cos(nx)\Big|_0^{\pi} - \frac{2}{\pi n^2}\int_0^{\pi}\cos(nx)\,dx = \frac{2(-1)^n}{n^2} - \frac{2}{\pi n^3}\underbrace{\sin(nx)\Big|_0^{\pi}}_{\parallel\, 0}$$

$$= \frac{2(-1)^n}{n^2}$$

$$b_n = -\frac{1}{\pi}\int_0^{\pi} \cancel{\sin}\ x^2 \sin nx\,dx = \frac{1}{n\pi} x^2\cos(nx)\Big|_0^{\pi} \mp \frac{2}{n\pi}\int_0^{\pi} x\cos nx\,dx$$

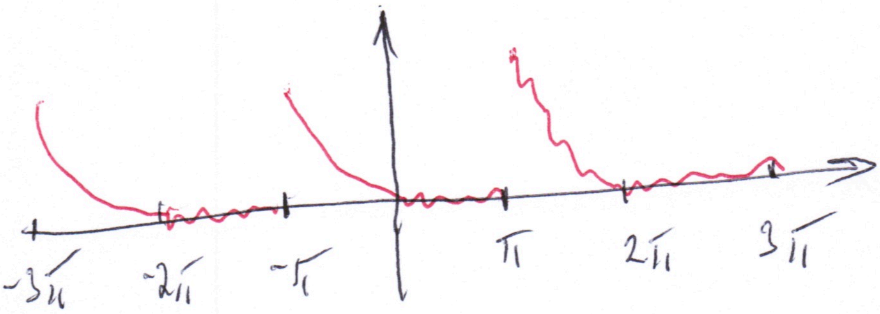
$$= \frac{\pi}{n}(-1)^n - \frac{2}{\pi n^2} x\sin nx\Big|_0^{\pi} + \frac{2}{\pi n^2}\int_0^{\pi}\sin nx\,dx =$$

$$= \frac{\pi}{n}(-1)^n - \frac{2}{\pi n^3}\cos(nx)\Big|_0^{\pi} = \boxed{\frac{\pi}{n}(-1)^n - \frac{2}{\pi n^3}\left((-1)^n - 1\right)}$$

Typeset by $\mathcal{AMS}$-TEX

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx) + \sum_{n=1}^{\infty}\left(\frac{\pi(-1)^n}{n} + \frac{2}{\pi n^3}\left(1-(-1)^n\right)\right)\sin(nx)$$

3)



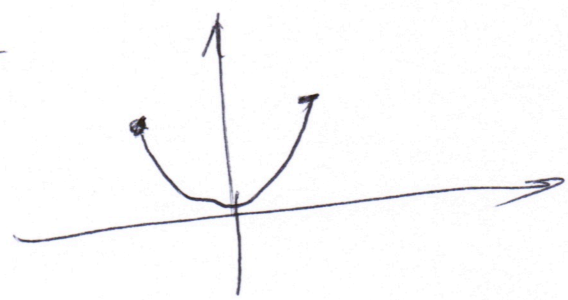
2) $x = 0$

$$0 = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \; ; \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$x = \pi$

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{1}{n^2} \; ; \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} .$$

4) $g(x) = \begin{cases} x^2/2 , & -\pi < x < \pi \\ \end{cases}$



2. (45 points) 1) Give the formulas connecting the coefficients of real and complex Fourier series of the same function.

2) Compute the complex Fourier series of the function from Problem 1 on the interval $-\pi < x < \pi$ by 2 methods:

(i) by the direct computation of the coefficients;

(ii) by the transformation of the coefficients of the real Fourier series from Problem 1 in the coefficients of the complex series.

Compare the results.

1) $a_n = c_n + c_{-n}$; $b_n = i(c_n - c_{-n})$

$$c_n = \frac{a_n - i b_n}{2}, \quad c_{-n} = \frac{a_n + i b_n}{2}$$

$$c_0 = \frac{a_0}{2}$$

2) (ii) $$c_n = \frac{(-1)^n}{n^2} + i\left(\frac{\pi(-1)^{n+1}}{2n} + \frac{1}{\pi n^3}\left((-1)^n - 1\right)\right).$$

$$c_0 = \frac{\pi^2}{6}$$

(i) $$c_n = \frac{1}{2\pi}\int_{-\pi}^{0} x^2 \exp(-inx)\,dx = \frac{i}{2n\pi} x^2 \exp(-inx)\Big|_{-\pi}^{0}$$

$$- \frac{i}{n\pi}\int_{-\pi}^{0} x \exp(-inx)\,dx = i\,\frac{\pi(-1)^{n+1}}{2n} + \frac{1}{n^2\pi} x \exp(-inx)\Big|_{-\pi}^{0}$$

$$- \frac{1}{\pi n^2}\int_{-\pi}^{0} \exp(-inx)\,dx = i\,\frac{\pi(-1)^{n+1}}{2n} + \frac{(-1)^n}{n^2} + i\,\frac{1}{\pi n^3}\left((-1)^n - 1\right)$$

The seme formula.

3. (45 points) Find the eigenfunctions of the BVP

$$y'' + \lambda y = 0, y(0) = 0, y'(1) = 0.$$

1) $\lambda > 0$, $\lambda = a^2$

$y_a(x) = c_1 \cos(ax) + c_2 \sin(ax)$

$y_a(0) = c_1 = 0 \Rightarrow y_a(x) = c \sin(ax)$

$y_a'(x) = ac\cos(ax)$, $y_a'(1) = 0 \Rightarrow \cos(a) = 0 \Rightarrow$

$a = \frac{\pi}{2}(2k+1)$, $k = 0, 1, 2, \ldots$

$y_k(x) = \sin\left(\frac{\pi}{2}(2k+1)\right)$, are eigen functions

with $\lambda_k = \frac{\pi^2}{4}(2k+1)^2$, $k = 0, 1, 2, \ldots$

2) $\lambda < 0$, $\lambda = -a^2$

$y_a = c_1 \cosh ax + c_2 \sinh ax$

$y_a(0) = 0 \Rightarrow c_1 = 0$, $y_a = c \sinh ax$

$y_a'(x) = ac\cosh(ax)$,

$y_a'(1) = 0 \Rightarrow \cosh(a) = 0$. Impossible; $\cosh(x) > 0$.

3) $\lambda = 0$, $y_0(x) = c_0 + c_1 x$.

$y_0(0) = 0 \Rightarrow c_0 = 0$, $y_0(x) = c_1 x$

$y_0'(x) = c_1$. $y_0'(1) = 0 \Rightarrow c_1 = 0$, $y \equiv 0$. Contradiction.

4. (25 points) Consider the BVP

$$xy'' + 4y' + \lambda y = 0, y(1) = 0, y'(2) = 0.$$

Put the equation in the self-adjoint (Sturm-Liouville) form and give the orthogonality condition of the eigenfunctions.

Is this problem regular? What kind BVP we can consider on the interval $0 < x < 1$?

$\mu(x) = x^3$

1st way - "to guess"

2nd way

$(\mu x)' = 4\mu$

$\mu' x + \mu = 4\mu$

$\frac{\mu'}{\mu} = \frac{3}{x}$

$(\ln \mu)' = \frac{3}{x}$

$\ln \mu = 3 \ln x$

$\mu = x^3$

$(x^4 y')' + \lambda x^3 = 0$

$r > 0, p > 0$ on $[1, 2]$

The problem is regular.

On $[0, 1]$ $r(0) = 0$

Singular problem (no conditions for $x = 0$).

5. (25 points) 1) Write down Legendre's equation and verify that the polynomials

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

are eigenfunctions.

2) Verify directly that P_0, P_1, P_2 are orthogonal on the interval $-1 < x < 1$ and compute their norms.

3) Find first 4 coefficients in the Fourier-Legendre series of the function $f(x) = x^3$.

$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$

2) $(P_0, P_1) = (P_1, P_2) = 0$ since we have $\int_{-1}^{1}$ of odd function

$(P_0, P_2) = \frac{1}{2}\int_{-1}^{1}(3x^2-1)\,dx = \frac{1}{2}(x^3 - x)\Big|_{-1}^{1} = 0.$

3) $x^3 = c_0 + c_1 P_1 + c_2 P_2 + c_3 P_3 + \dots$

$c_0 = c_2 = 0$ (odd functions) $\quad \|P_n\|^2 = \frac{2}{2n+1}$

$c_1 = \frac{3}{2}\int_{-1}^{1} x^4\,dx = \frac{3}{10}x^5\Big|_{-1}^{1} = \frac{3}{5}.$

$c_3 = \frac{7}{2}\cdot\frac{1}{2}\int_{-1}^{1}(5x^6 - 3x^4)\,dx = \frac{7}{4}\left(\cancel{x^5 - 2x^3}\right)$

$= \frac{7}{4}\left(\frac{5}{7}x^7 - \frac{3}{5}x^5\right)\Big|_{-1}^{1} = \frac{2}{5}.$

MATH 421.08. ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. MIDTERM 2

1. (60 points) 1) Find the Fourier series on the interval $-2 < x < 2$ of the function $f(x) = x^2$ for $0 < x < 2$ and $f(x) = 0$ for $-2 < x < 0$.

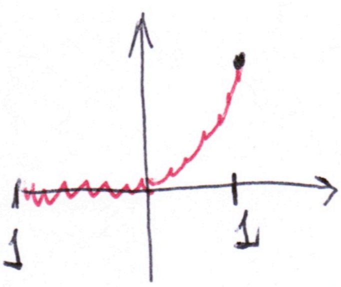
2) Describe the function which this series represents on the whole line $-\infty < x < \infty$.

3) Specify the numerical series corresponding to the points $x = 0$ and $x = 2$.

4) Let us keep in the Fourier series only terms with Cosines. Which function will represent this new series? Give complete explanations.

$p = 2$

$$a_0 = \frac{1}{2}\int_0^2 x^2\,dx = \frac{8}{6} = \frac{4}{3};$$



$$a_n = \frac{1}{2}\int_0^2 x^2 \cos\left(\frac{n\pi x}{2}\right)dx =$$

$$= \underbrace{\frac{1}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2}_{=0} - \frac{2}{n\pi}\int_0^2 x \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{4}{(n\pi)^2}\, x\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 -$$

$$- \frac{4}{(n\pi)^2}\underbrace{\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx}_{\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2 = 0} = \boxed{\frac{8(-1)^n}{(n\pi)^2}}.$$

$$b_n = \frac{1}{2}\int_0^2 x^2 \sin\left(\frac{n\pi x}{2}\right)dx = -\frac{1}{n\pi}x^2\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 +$$

$$+ \frac{2}{n\pi}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx =$$

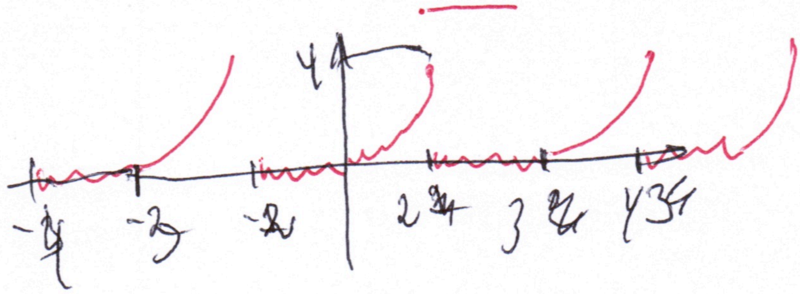
$$= \frac{4(-1)^{n+1}}{\pi n} + \frac{8}{\pi^3 n^3}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 =$$

$$= \boxed{\frac{4(-1)^{n+1}}{\pi n} + \frac{8}{\pi^3 n^3}\left((-1)^n - 1\right).}$$

$$f(x) = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right) +$$

$$+ \frac{4}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}\pi^2}{n} + \frac{2\left((-1)^n - 1\right)}{n^3}\right)\sin\left(\frac{n\pi x}{2}\right).$$

3)



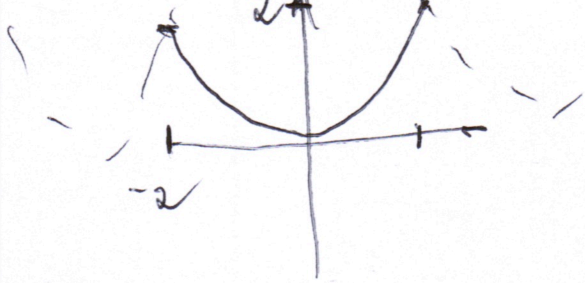
2) $x = 0$

$$0 = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \quad \Big| \quad \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$x = 2$

$$2 = \frac{2}{3} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \quad \Big| \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

4)



$$f_e(x) = \frac{x^2}{2}, \quad -2 \le x \le 2.$$

$$f_e(x) = \frac{1}{2}\left(f(x) + f(-x)\right)$$

2. (45 points) 1) Give the formulas connecting the coefficients of real and complex Fourier series of the same function.

2) Compute the complex Fourier series of the function from Problem 1 on the interval $-2 < x < 2$ by 2 methods:

(i) by the direct computation of the coefficients;

(ii) by the transformation of the coefficients of the real Fourier series from Problem 1 in the coefficients of the complex series.

Compare the results.

1) $a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n});$

$$c_n = \frac{a_n - ib_n}{2}, \ -\infty < n < \infty; \quad c_0 = \frac{a_0}{2}$$

2) (ii) $$c_n = \frac{4(-1)^n}{\pi^2 n^2} + i\left(\frac{2(-1)^n}{\pi n} + \frac{4(1-(-1)^n)}{\pi^3 n^3}\right)$$

$$c_0 = \frac{a_0}{2} = \frac{2}{3}$$

(i) $$c_n = \frac{1}{4}\int_0^2 x^2 \exp\left(-i\frac{n\pi x}{2}\right)dx = \frac{i}{2n\pi} x^2 \exp\left(-\frac{in\pi}{x}\right)\Big|_0^2$$

$$-\frac{i}{n\pi}\int_0^2 x \exp\left(-i\frac{n\pi x}{2}\right)dx = \frac{2(-1)^n}{\pi n} i + \frac{2}{n^2\pi^2} x \exp\left(-\frac{in\pi}{2}x\right)\Big|_0^2$$

$$\cancel{+ \frac{2}{n^3\pi^3} - \frac{2}{n^3\pi^3}} - \frac{2}{n^2\pi^2}\int_0^2 \exp\left(-i\frac{n\pi x}{2}\right)dx =$$

$$= \frac{2(-1)^n}{\pi n} i + \frac{4(-1)^n}{n^2\pi^2} - i\frac{4}{\pi^3 n^3} \exp\left(-\frac{in\pi x}{2}\right)\Big|_0^2 =$$

$$= \boxed{\frac{2(-1)^n}{\pi n}\, i + \frac{4(-1)^n}{n^2\pi^2} + \frac{4(1-(-1)^n)}{\pi^3 n^3}\, i}\,.$$

The same formula!

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(i\, \frac{n\pi x}{a}\right)$$

3. (45 points) Find the eigenfunctions of the BVP

$\lambda > 0$

$$y'' + \lambda y = 0, y'(0) = 0, y(1) = 0.$$

1) $\lambda = a^2, a > 0$

$y_a(x) = C_1 \cos ax + C_2 \sin ax,$

$y_a'(x) = -aC_1 \sin ax + aC_2 \cos ax$

$y_a'(0) = 0 \Rightarrow C_2 = 0, \ y_a(x) = c \cos ax, \ y_a'(x) = -ac \sin ax$

$y_a(1) = c \cos a = 0 \Rightarrow a = \frac{\pi}{2}(2k+1), \ k = 0, 1, 2, \ldots$

$y_k(x) = \cos\left(\frac{\pi}{2}(2k+1)\right)$ are eigen functions

with the eigen values $\lambda_k = \frac{\pi^2}{4}(2k+1)^2, \ k = 0, 1, 2, \ldots$

2) $\lambda < 0, \ \lambda = -a^2$

$y_a(x) = c_1 \cosh ax + c_2 \sin ax,$

$y_a'(x) = ac_1 \sinh ax + ac_2 \cosh ax;$

$y_a'(0) = 0 \Rightarrow c_2 = 0, \ y_a(x) = c \cosh ax,$

$y_a(1) = 0 \Rightarrow \cosh a = 0$, but $\cosh x > 0$.

Contradiction. No eigen values $\lambda < 0$.

3) $\lambda = 0, \ y_0 = C_1 x + C_0.$

$y'(x) = C_1; \ y'(0) = 0 \Rightarrow C_1 = 0, \ y_0(x) = c.$

$y_0(1) = 0 \Rightarrow c = 0, \ y \equiv 0.$ $\lambda = 0$ isn't an eigen value.

4. (25 points) Consider the BVP

$$y'' + (\cos x)y' + (\lambda \sin x)y = 0, \quad y(\pi/4) = 0, y'(\pi/2) = 0.$$

Put the equation in the self-adjoint (Sturm-Liouville) form and give the orthogonality condition of the eigenfunctions.

Is this problem regular?

$\mu' = \mu \cos x,$

$\frac{\mu'}{\mu} = \cos x,$

$(\ln \mu)' = \cos x,$

$\mu = e^{\sin x}.$

$(e^{\sin x} y')' + \lambda \sin x \, e^{\sin x} y = 0$

$r = e^{\sin x}, \; p = \sin x \, e^{\sin x}, \; q = 0.$

Eigen functions are orthogonal with the weight $p(x)$.

r, p are positive on ~~[...]~~ $[\frac{\pi}{4}, \frac{\pi}{2})$

are BVP is regular

5. (25 points) 1) Write down Legendre's equation. Verify that the polynomials

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

are eigenfunctions.

2) Verify directly that P_0, P_1, P_2 are orthogonal on the interval $-1 < x < 1$.

3) Find first 4 coefficients in the Fourier-Legendre series of the function $f(x) = x^4$.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

2) We need to verify only that

$$(P_0, P_2) = \frac{1}{2}\int_{-1}^{1}(3x^2-1)\,dx$$

$$= \frac{1}{2}(x^3 - x)\Big|_{-1}^{1} = 0.$$

$(P_0, P_1) = (P_1, P_2) = 0$ since we integrate on $[-1,1]$ odd functions.

3) $x^4 = c_0 + c_1 P_1 + c_2 P_2 + c_3 P_3 + \dots$

$\|P_n\|^2 = \frac{2}{2n+1}$; $\quad c_1 = c_3 = 0$ ($\int_{-1}^{1}$ of odd functions).

$$c_0 = \frac{1}{2}\int_{-1}^{1} x^4\,dx = \frac{1}{5};$$

$$c_2 = \frac{5}{2}\cdot\frac{1}{2}\int_{-1}^{1}(3x^2-1)x^4\,dx = \frac{5}{2}\cdot\left(\frac{3x^7}{7} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{4}{7}.$$