MATH 421.03. ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. MIDTERM 1

1. (45 points)

Find given Laplace transforms

$\mathcal{L}\{te^{-t}\sinh(3t)\}$;

$\mathcal{L}\{\mathcal{U}(t-2)t^2e^{-t}\}$;

$\mathcal{L}\{e^{2t}\delta(t-3)\}$.

$$1)\ \mathcal{L}(te^{-t}\sinh(3t)) = \cancel{A}\ \frac{6(s+1)}{(s^2+2s-8)}$$

$$\frac{3}{s^2-9} \Rightarrow +\frac{6s}{(s^2-9)^2} \Rightarrow +\frac{6(s+1)}{((s+1)^2-9)^2}$$

$$2)\ \mathcal{L}(\mathcal{U}(t-2)t^2e^{-t}) = e^{-2s}\mathcal{L}((t+2)^2e^{-t-2})$$

$g(t) = t^2e^{-t}$

$a=2$

$$= e^{-2s-2}\mathcal{L}(e^{-t}(t^2+4t+4))$$

$$= e^{-2s-2}\left(\frac{2}{(s+1)^3}+\frac{4}{(s+1)^2}+\frac{4}{s+1}\right),$$

$$3)\ \mathcal{L}\{e^{2t}\delta(t-3)\} = e^{-\cancel{s}3(s-2)}$$

$$\mathcal{L}(\delta(t-3)) = \cancel{e^{-2s}}\ e^{-3s}.$$

Other solution: $e^{2t}\delta(t-3) = e^6\delta(t-3)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. (45 points)
Find given inverse Laplace transforms

$$\mathcal{L}^{-1}\left(\frac{s^2}{(s^2+9)(s^2-4)}\right) = \frac{3}{13}\sin 3t + \frac{2}{13}\sinh 2t.$$

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-6s+11}\right) = \frac{\sqrt{2}}{2}e^{3(t-3)}\sin(\sqrt{2}(t-3))\,u(t-$$

$$\mathcal{L}^{-1}\left(\frac{e^{-s}(s^2-1)}{s^2-2s+2}\right) = \delta(t-1) + u(t-1)e^{t-1}(2\cos(t-1) - \sin(t-1))$$

1) $$\frac{s^2}{(s^2+9)(s^2-4)} = \frac{1}{13}\left(\frac{9}{s^2+9} + \frac{4}{s^2-4}\right)$$

2) $$\frac{e^{-3s}}{(s-3)^2+2}$$

3) $$e^{-s}\left(1 + \frac{2(s-1)-1}{(s-1)^2+1}\right)$$

3. (40 points) Solve given initial-value problem using the Laplace transform
$y'' - 2y' + 5y = e^t t, y(0) = -1, y'(0) = -2;$

$$(s^2 - 2s + 5)Y = \frac{1}{(s-1)^2} - s$$

$$[(s-1)^2 + 4]Y = \frac{1}{(s-1)^2} - (s-1) - 1$$

$$Y = \frac{1}{(s-1)^2[(s-1)^2+4]} - \frac{s-1}{(s-1)^2+4} - \frac{1}{[(s-1)^2+4]}$$

$$y = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2} - \frac{1}{(s-1)^2+4}\right) - $$

$$-e^t\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

$$y(t) = e^t\left(\frac{t}{4} - \frac{1}{8}\sin 2t - \cos 2t - \frac{1}{2}\sin 2t\right)$$

$$= e^t\left(\frac{t}{4} - \frac{5}{8}\sin 2t - \cos 2t\right).$$

$y(0) = -1.$

4. (35 points) Solve given integral equation
$y(t) = te^{-t} + \int_0^t y(\tau)\sin(2(t-\tau))d\tau.$

$$Y = \frac{1}{(s+1)^2} + \frac{2}{s^2+4}Y$$

$$Y\left(1 - \frac{2}{s^2+4}\right) = \frac{1}{(s+1)^2}$$

$$Y = \frac{s^2+4}{(s+1)^2(s^2+2)} = \frac{as+B}{(s+1)^2} + \frac{Cs+D}{s^2+2}$$

$$(as+B)(s^2+2) + (Cs+D)(s+1)^2 = s^2+4$$

$$\begin{array}{c|l}
s^3 & a+C=0 \\
s^2 & B+2C+D=\cancel{4}\,1 \\
s & 2a+C+2D=0 \\
1 & 2B+D=4
\end{array}$$

$C=-a$

$a+2D=0$

$D=4-2B$

$2C-B=-3$

$a=\frac{4}{9}$

$C=-\frac{4}{9}$

$D=-\frac{2}{9}$

$B=\frac{19}{9}$

$$y = \frac{1}{9}\left(4e^{-t} + 15te^{-t} - 4\cos(\sqrt{2}t) - \sqrt{2}\sin\sqrt{2}t\right).$$

5. (35 points) a) State the theorem on the Laplace transform of the convolution. b) Apply this result for the computation of the convolution

$$\int_0^t e^{\tau} \cos(t-\tau)d\tau. = x(t)$$

a) $\mathcal{L}(f * g) = \mathcal{L}f \, \mathcal{L}g$.

b) $\mathcal{L}x = \frac{1}{s-1} \frac{s}{s^2+1} = \frac{a}{s-1} + \frac{Bs+C}{s^2+1} =$

$= \frac{1}{2}\left(\frac{1}{s-1} - \frac{s-1}{s^2+1}\right)$

$x(t) = \frac{1}{2}(e^t - \cos t + \sin t)$.

MATH 421.08 ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. MIDTERM 1

1. (45 points)

Find given Laplace transforms

$\mathcal{L}\{te^{-3t}\cos(3t)\}$; $\Rightarrow$ $\dfrac{s(s+6)}{(s^2+6s+18)^2}$

$\mathcal{L}\{\mathcal{U}(t-\frac{\pi}{2})\cos(3t)e^{2t}\}$;

$\mathcal{L}\{t^2\delta(t-2)\}$.

1) $\dfrac{s}{s^2+9} \Rightarrow \dfrac{s^2-9}{(s^2+9)^2} \Rightarrow \dfrac{-9+(s+3)^2}{((s+3)^2+9)^2} = \dfrac{s(s+6)}{(s^2+6s+18)^2}$

2) $g(t) = \cos 3t \ \ e^{2t}$

$a = \pi/2$

$e^{-\pi/2 s} \mathcal{L}(\cos 3(t+\frac{\pi}{2}))e^{2(t+\frac{\pi}{2})}) =$

$= e^{(\pi - \pi s/2)} \mathcal{L}(e^{2t} \sin 3t) = \dfrac{3e^{(\pi-\pi s/2)}}{(s-2)^2+9}$

3) $(e^{-2s})'' = 4e^{-2s}$.

Other solution

$t^2\delta(t-2) = 4\delta(t-2)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. (45 points)
Find given inverse Laplace transforms

$$1)\ \mathcal{L}^{-1}\left(\frac{s^2}{(s^2-1)(s^2+4)}\right);$$

$$2)\ \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-5s-6}\right).$$

$$3)\ \mathcal{L}^{-1}\left(\frac{e^{-s}(s^2+2)}{s^2-4s+2}\right).$$

$$1)\ \frac{1}{5}\left(\cancel{4}\,\mathcal{L}^{-1}\left(\frac{1}{s^2-1}+\frac{4}{s^2+4}\right)\right)=\frac{1}{5}\left(\sinh t+2\sin 2t\right)$$

$$2)\ \mathcal{L}^{-1}\left(\frac{e^{-3s}}{(s-\frac{5}{2})^2-\frac{49}{4}}\right)=\frac{1}{7}u(t-3)\left(e^{6(t-3)}-e^{-t+3}\right)$$

$$=\frac{2}{7}u(t-3)e^{5/2(t-3)}\sinh\frac{7}{2}(t-3).$$

$$3)\ \mathcal{L}^{-1}\left(e^{-s}\left(1+\frac{4(s-2)+8}{(s-2)^2-2}\right)\right)=\delta(t-1)+$$

$$\cancel{+4e^{2(t-1)}(4\cosh}$$

$$+4e^{2(t-1)}\left(\cosh(\sqrt{2}(t-1))+\sqrt{2}\sinh(\sqrt{2}(t-1))\right).$$

3. (40 points) Solve given initial-value problem using the Laplace transform
$y'' + 4y' + 3y = e^{-2t}\cos t, y(0) = 1, y'(0) = -2;$

$$(s^2+4s+3)Y = \frac{s+2}{(s+2)^2+1} + s + 2$$

$$Y = \frac{s+2}{((s+2)^2+1)((s+2)^2-1)} + \frac{s+2}{(s+2)^2-1}$$

$s+2=x$ $$\frac{x}{(x^2+1)(x^2-1)} = \frac{1}{2}\left(\frac{-x}{x^2+1} + \frac{x}{x^2-1}\right)$$

$$Y = -\frac{1}{2}\frac{s+2}{(s+2)^2+1} + \frac{3}{2}\frac{s+2}{(s+2)^2-1}$$

$$y(t) = e^{-2t}\left(\frac{3}{2}\cosh t - \frac{1}{2}\cos t\right)$$

$y(0)=1$

4. (35 points) Solve given integro-differential equation
$y'(t) = \sin(2t) + 4\int_0^t y(\tau)d\tau, y(0) = -2.$

$$sY = \frac{2}{s^2+4} + \frac{4}{s}Y - 2$$

$$\left(s - \frac{4}{s}\right)Y = \frac{-2s^2-6}{s^2+4}$$

$$Y = \frac{-2s(s^2+3)}{(s^2+4)(s^2-4)} = -2s\left(\frac{a}{s^2+4} + \frac{B}{s^2-4}\right)$$

$$a + B = 1 \qquad a = \frac{1}{8}$$
$$-4a + 4B = 3 \qquad b = \frac{7}{8}$$

$$y = -2\left(\cancel{\cos 2t}\ a\cos 2t + B\cosh 2t\right) =$$
$$= -\frac{1}{4}\left(\cos 2t + 7\cosh 2t\right).$$

$$y(0) = -2.$$

5. (35 points) Solve the initial-value problem for given system of differential equations using the Laplace transform

$$x' = 2x + y, y' = x + 2y, x(0) = 1, y(0) = -2.$$

$X = \mathcal{L}x, \quad Y = \mathcal{L}y$

$X(s-2) - Y = 1$

$-X + Y(s-2) = -2$

$((s-2)^2 - 1)Y = 1 - 2(s-2)$

$$Y = \frac{-2(s-2)+1}{(s-2)^2 - 1}$$

$y = e^{2t}(-2\cosh t + \sinh t)$

$= -\frac{1}{2}e^{3t} - \frac{3}{2}e^{t}$

$x = y' - 2y = -\frac{1}{2}e^{3t} + \frac{3}{2}e^{t} =$

$= e^{2t}(\cosh t - 2\sinh t)$