Lecture 9.

Part 2.

Orthogonal functions and Fourier Series.

Ch. 12.

Geometrical view on functions and series.

- Linear space of functions (infinite dimensional) ~~over~~ $[a, b]$.
- functons $\Leftrightarrow$ vectors
- Series $\Leftrightarrow$ expansion on a base

Orthogonality:

The inner product

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx$$

$$f_1 \perp f_2 \Leftrightarrow (f_1, f_2) = 0$$

orthogonal

$$\|f\|^2 = (f, f) = \int_a^b f^2(x)\, dx$$

norm

Orthogonal set $\{\varphi_1(x), \dots \varphi_n(x), \dots\}$

$\varphi_i \perp \varphi_j$, $i \neq j$.

Examples.

1) $\{\sin x, \sin 2x, \dots\}$ is orthog. set on $[-\pi, \pi]$

$$(\varphi_n, \varphi_m) = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx =$$

$n \neq m$

$$\frac{1}{2}\int_{-\pi}^{\pi}\left(\cos(n-m)x - \cos(n+m)x\right)dx$$

$$= \frac{1}{2}\left(\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right)\Bigg|_{-\pi}^{\pi}$$

$= 0$ $\quad (\sin k\pi = 0)$

if $n \neq m$

What is for $n = m$?

$$\|\varphi_n\|^2 = (\varphi_n, \varphi_n) = \int_{-\pi}^{\pi} \sin^2(nx)\,dx =$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left(1 - \cos(2nx)\right)dx$$

$= \pi$ $\qquad \|\varphi_n\| = \sqrt{\pi}$

9.1

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$

$$\cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$

$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

2) $\{\cos nx,\ n=0,1,2,\dots\} = \{\psi_n\}$

$$(\cos nx, \cos mx) = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$$

$n \neq m$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (\cos(n-m)x + \cos(n+m)x)\,dx$$

$$= 0 \quad (n \neq m)$$

$\|\psi_0\|^2 = 2\pi$ $\qquad$ $\|1\| = \sqrt{2\pi}$

$\|\psi_n\|^2 = \pi$ $\qquad$ $\|\cos nx\| = \sqrt{\pi}$

$n \neq 0$ $\qquad$ $n \neq 0$

3) $\varphi_n \perp \psi_m$

$$(\varphi_n^*, \psi_m) = \int_{-\pi}^{\pi} \sin nx \cos mx\,dx = 0$$

We can integrate as above or ...

We can remark that we integrate an odd function along a symmetric interval.