MATH 421.08. ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 2

1. (90 points)

a) Find given Laplace transforms
$\mathcal{L}\{3t\sin(2t)\}$;
$\mathcal{L}\{\mathcal{U}(t-1)(t-1)^2\}$.
$\mathcal{L}\{\mathcal{U}(t-1)t^2\}$.

b) Find given inverse Laplace transform
$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+4s+5}\right)$;
$\mathcal{L}^{-1}\left(\frac{4s^2-5}{2s^2-8s+4}\right)$.

a) 1. $\mathcal{L}(3t\sin 2t) = -6\left(\frac{1}{s^2+4}\right)' = \frac{12s}{(s^2+4)^2}$.

2. $\mathcal{L}\{\mathcal{U}(t-1)(t-1)^2\} = e^{-s}\mathcal{L}(t^2) = \frac{2e^{-s}}{s^3}$.

3. $\mathcal{L}(\mathcal{U}(t-1)t^2) = e^{-s}\mathcal{L}(t^2+t+1) =$

$a=1,\ g(t)=t^2$. $= e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$.

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b) $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+4s+5}\right) = u(t-2)\,e^{-2(t-2)}\sin(t-2).$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+4s+5}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+1}\right) = e^{-2t}\sin t$$

$$\mathcal{L}^{-1}\left(\frac{4s^2-5}{2s^2-8s+4}\right) = \mathcal{L}^{-1}\left(2+\frac{16s-13}{2s^2-8s+4}\right)$$

$$= 2\delta(t) + \mathcal{L}^{-1}\left(\frac{16(s-2)+19}{2\left((s-2)^2-2\right)}\right) =$$

$$= 2\delta(t) + e^{2t}\left(8\cosh(\sqrt{2}t) + \frac{19}{2\sqrt{2}}\sinh(\sqrt{2}t)\right)$$

2. (60 points) Solve given initial-value problem using the Laplace transform
$y'' - 4y' + 5y = e^t, y(0) = -1, y'(0) = 2;$

$$(s^2 - 4s + 5)Y = \frac{1}{s-1} - s + 6$$

$$\left((s-2)^2 + 1\right)Y = \frac{1}{s-1} - (s-2) + 4$$

$$Y = \frac{1}{(s-1)\left((s-2)^2+1\right)} - \frac{s-2}{(s-2)^2+1} + \frac{4}{(s-2)^2+1}$$

$$\frac{1}{(s-1)\left((s-2)^2+1\right)} = \frac{a}{s-1} + \frac{B(s-2)+C}{(s-2)^2+1}$$

$$y(t) = ae^t + e^{2t}\left((B-1)\cos t + (C+4)\sin t\right)$$

$$1 = a\left((s-2)^2+1\right) + (s-1)\left(B(s-2)+c\right)$$

$$\begin{array}{l|l|l}
s^2 & a+B=0 & B=-a \\
s & -4a-3B+C=0 & \\
1 & 5a+2B-C=1 & a-B=1
\end{array}$$

$a = \frac{1}{2}, B = -\frac{1}{2}, C = \frac{1}{2}$

$$y(t) = \frac{e^t}{2} + e^{2t}\left(-\frac{3}{2}\cos t + \frac{9}{2}\sin t\right), \quad y(0) = -1$$

3. (50 points) Solve given integral equation
$y(t) = e^t + 2\int_0^t \sin(t-\tau)y(\tau)d\tau$.

$$Y = \frac{1}{s-1} + 2\frac{Y}{s^2+1}$$

$$\left(1 - \frac{2}{s^2+1}\right)Y = \frac{1}{s-1}$$

$$\frac{s^2-1}{s^2+1}Y = \frac{1}{s-1}$$

$$Y = \frac{s^2+1}{(s^2-1)(s-1)} = \frac{s^2+1}{(s-1)^2(s+1)}$$

$$\frac{s^2+1}{(s-1)^2(s+1)} = \frac{a(s-1)+B}{(s-1)^2} + \frac{C}{s+1}$$

$$a(s^2-1) + B(s+1) + C(s-1)^2 = s^2+1$$

$$\begin{array}{l|l} s^2 & a + C = 1 \\ s & B - 2C = 0 \\ 1 & -a + B + C = 1 \end{array}$$

$a = 1 - C$
$B = 2C$
$4C = 2$

$C = 1/2$
$B = 1$
$a = 1/2$

$$y(t) = Ce^{-t} + e^t(a + Bt)$$

$$= \frac{e^{-t}}{2} + e^t\left(\frac{1}{2} + t\right) = te^t + \cosh t.$$

MATH 421.03 ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 2

1. (90 points)

a) Find given Laplace transforms

$\mathcal{L}\{2t\cos(2t)\}$;

$\mathcal{L}\{\mathcal{U}(t-\pi/2)\sin(t-\pi/2)\}$;

$\mathcal{L}\{\mathcal{U}(t-1)t^2\}$.

b) Find given inverse Laplace transform

$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-3s+2}\right)$;

$\mathcal{L}^{-1}\left(\frac{3s^2-2s}{s^2-6s+13}\right)$.

a) 1.) $\mathcal{L}\{2t\cos(2t)\} = -2\,\mathcal{L}(\cos 2t)' = -2\left(\frac{s}{s^2+4}\right)'$

$$= -2\,\frac{s^2+4-s^2}{(s^2+4)^2} = \frac{-2s^2-8}{(s^2+4)^2}$$

2) $\mathcal{L}\{\mathcal{U}(t-\frac{\pi}{2})\sin(t-\pi/2)\} = e^{-\frac{\pi s}{2}}\mathcal{L}(\sin t) =$

$$= \frac{e^{-\pi s/2}}{s^2+1},$$

3) $\mathcal{L}\{\mathcal{U}(t-1)t^2\} = e^{-s}\mathcal{L}((t+1)^2) =$

$a=1,\ g(t)=t^2$

$$= e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$$

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2. (60 points) Solve given initial-value problems using the Laplace transform
$y'' - 4y' + 3y = 2, y(0) = 2, y'(0) = -2;$

$$(s^2-4s+3)Y = \frac{2}{s} + sy(0) + y'(0) - 4y(0) =$$

$$= \frac{2}{s} + 2s - 10.$$

$$(s-3)(s-1)Y = \frac{2}{s} + 2(s-3) - 4.$$

$$Y = \frac{2-4s}{s(s-3)(s-1)} + \frac{2}{s-1}$$

$$\frac{2-4s}{s(s-3)(s-1)} = \frac{a}{s} + \frac{B}{s-3} + \frac{C}{s-1}$$

$$2-4s = a(s-2)(s-1) + B\,s(s-1) + C\,s(s-3)$$

$$\begin{array}{l|l} s=0 & 2 = 3a, \quad a = \frac{2}{3} \\ s=3 & -10 = 6B, \quad B = -5/3 \\ s=1 & -2 = -2C, \quad C = 1 \end{array}$$

$y(0) = 2$

$$y(t) = a + Be^{3t} + (C+2)e^t$$

$$= 3e^t - \frac{5}{3}e^{3t} + \frac{2}{3}$$

$$= e^{2t}\left(\frac{4}{3}\cosh t - \frac{14}{3}\sinh t\right) + \frac{2}{3}$$

3. (50 points) Solve given integral equation
$y'(t) = e^t + \int_0^t y(\tau)d\tau, y(0) = 1.$

$Y = \mathcal{L}y$

$$sY = \frac{1}{s-1} + \frac{Y}{s} + 1$$

$$\left(s - \frac{1}{s}\right)Y = \frac{1}{s-1} + 1 = \frac{s}{s-1}$$

$$\left(\frac{s^2-1}{s}\right)Y = \frac{s^2}{(s^2-1)(s-1)} = \frac{s^2}{(s-1)^2(s+1)}$$

$$\frac{a(s-1)+B}{(s-1)^2} + \frac{C}{s+1} = \frac{s^2}{(s-1)^2(s+1)}$$

$$a(s^2-1) + B(s+1) + C(s-1)^2 = s^2$$

$a + C = 1$ $\quad$ $C = -\frac{1}{2},\ B = -1,\ a = \frac{3}{2}$

$B - 2C = 0$

$a + B + C = 0$

$$y = ae^t + Be^t \cdot t + Ce^{-t}$$

$$= \frac{3}{2}e^t - te^t - \frac{1}{2}e^{-t}.$$

$y(0) = 1$