MATH 421.03 ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 1

1. (75 points)

a) Find given Laplace transforms

$\mathcal{L}\{(t-1)^2\}$;

$\mathcal{L}\{e^{2t}\cos 3t\}$;

$\mathcal{L}\{\mathcal{U}(t-1)e^{2t}\}$.

b)Give the formula representing $\mathcal{L}\{e^{at}f\}$ through $\mathcal{L}f$ and prove it.

a) $\mathcal{L}((t-1)^2) = \mathcal{L}(t^2) - 2\mathcal{L}(t) + \mathcal{L}(1) = \frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}$,

$\mathcal{L}(e^{2t}\cos 3t) = \frac{s-2}{(s-2)^2+9} = \frac{s-2}{s^2-4s+13}$.

$\mathcal{L}(\mathcal{U}(t-1)e^{-2t}) = e^{-s}e^{2}\frac{1}{s-2} = \frac{e^{2-s}}{s-2}$

$g(t) = e^{2t}$, $a=1$

$g(t+1) = e^2 e^{2t}$

b) $\mathcal{L}(e^{at}f) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \mathcal{L}f(s-a)$,

$a>0$.

2. (75 points) Find given inverse Laplace transforms

$$\mathcal{L}^{-1}\left(\frac{s^2}{(s+1)(s-2)(s+3)}\right) = \frac{1}{6}e^{-t} + \frac{4}{15}e^{2t} + \frac{9}{10}e^{-3t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-4s+3}\right) = e^{2t}\sinh t = \frac{1}{2}(e^{3t}-e^{t}),$$

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^4}\right) = \frac{1}{6}u(t-2)(t-2)^3.$$

$$\mathcal{L}^{-1}\left(\frac{2s+1}{s^2+2s+5}\right) = \cancel{e^{2t}}\, e^{-t}\left(2\cos 2t - \frac{1}{2}\sin 2t\right)$$

1) $\dfrac{s^2}{(s+1)(s-2)(s+3)} = \dfrac{a}{s+1} + \dfrac{B}{s-2} + \dfrac{C}{s+3}$

$a(s-2)(s+3) + B(s+1)(s+3) + c(s+1)(s-2) = s^2$

$s=1 \Rightarrow a(-6)=1, \quad a=-\frac{1}{6}$

$s=2 \Rightarrow B(3\cdot5)=4, \quad B=\frac{4}{15}$

$s=-3 \Rightarrow C(-2\cdot-5)=9, \quad C=\frac{9}{10}.$

2) $\dfrac{1}{s^2-4s+3} = \dfrac{1}{(s-2)^2-1} = \dfrac{1}{(s-3)(s-1)} = \dfrac{1}{2}\left(\dfrac{1}{s-3} - \dfrac{1}{s-1}\right).$

4) $\dfrac{2s+1}{s^2+2s+5} = \dfrac{2(s+1)-1}{(s+1)^2+4}$

3. (50 points) Solve given initial-value problem using the Laplace transform
$y'' - y' = 2t, y(0) = 1, y'(0) = 1.$

$$Y = \mathcal{L}y$$

$$(s^2 - s)Y = \frac{2}{s^2} + sy(0) + y'(0) - y(0)$$

$$s(s-1)Y = \frac{2}{s^2} + s + 1 - 1 = \frac{2}{s^2} + s$$

$$Y = \frac{2}{s^3(s-1)} + \frac{1}{s-1}, \quad y = e^t + 2\mathcal{L}^{-1}\left(\frac{1}{s^3(s-1)}\right)$$

$$\frac{1}{s^3(s-1)} = \frac{as^2 + Bs + C}{s^3} + \frac{D}{s-1}$$

$$(as^2 + Bs + C)(s-1) + Ds^3 = 1$$

s^3	$a + D = 0$	$C = -1$
s^2	$-a + B = 0$	$B = -1$
s	$-B + C = 0$	$a = -1$
1	$-C = 1$	$D = 1$

$$\cancel{y(t) = at^2 + Bt + C + (2D+1)e^t}$$

$$\cancel{= -t^2 - t}$$

$$y(t) = Ct^2 + 2Bt + 2a + (2D+1)e^t$$

$$= -t^2 - 2t - 2 + 3e^t; \quad y(0) = 1; \; y'(t) = -2t - 2 + 3e^t$$

$$y'(0) = 1$$

MATH 421.08 ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 1

1. (75 points)

a) Find given Laplace transforms

$\mathcal{L}\{(1-e^t)^2\}$;

$\mathcal{L}\{e^{-2t}t^3\}$;

$\mathcal{L}\{\mathcal{U}(t-2)\sin 2t\}$.

b)Give the formula representing $\mathcal{L}f'$ through $\mathcal{L}f$ and prove it.

a) 1. $\mathcal{L}\{(1-e^t)^2\} = \mathcal{L}(1-2e^t+e^{2t}) =$

$= \frac{1}{s} - \frac{2}{s-1} + \frac{1}{s-2}.$

2. $\mathcal{L}(e^{-2t}t^3) = \frac{6}{(s+2)^4}.$

3. $\mathcal{L}(\mathcal{U}(t-2)\sin 2t) = e^{-2s}\left(\frac{2\cos 4}{s^2+4} + \frac{\sin(4)\cdot s}{s^2+4}\right).$

$a=2,\ g(t)=\sin 2t$

$g(t+2) = \sin(2t+4) = \sin 2t\cos 4 + \cos 2t \sin 4$

b) $\mathcal{L}f' = s\mathcal{L}f - f(0)$

$\mathcal{L}f' = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt$

$= -f(0) + s\mathcal{L}f.$

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2. (75 points) Find given inverse transforms

$$\mathcal{L}^{-1}\left(\frac{s}{(s-1)(s+2)(s-3)}\right) = -\frac{e^t}{6} - \frac{2e^{-2t}}{15} + \frac{3}{10}e^{3t};$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-4)^2}\right) = e^{4t}t;$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-s+5}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-\frac{1}{2})^2+\frac{19}{4}}\right) = \frac{2}{\sqrt{19}}e^{t/2}\sin\left(\frac{\sqrt{19}}{2}t\right)$$

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+4}\right) = \frac{1}{2}u(t-2)\sin(2(t-2)).$$

$$\frac{s}{(s-1)(s+2)(s-3)} = \frac{a}{s-1} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$a(s+2)(s-3) + B(s-1)(s-3) + C(s-1)(s+2) = s$$

$$\begin{array}{l|l} s=1 & -6a = 1 \\ s=-2 & 15B = -2 \\ s=3 & 10C = 3 \end{array} \qquad \begin{array}{l} a = -\frac{1}{6} \\ B = -\frac{2}{15} \\ C = \frac{3}{10} \end{array}$$

3. (50 points) Solve given initial-value problem using the Laplace transform
$y' - 2y = e^{3t}, y(0) = 2;$

$Y = \mathcal{L}y$

$(s-2)Y = \frac{1}{s-3} + y(0)$

$(s-2)Y = \frac{1}{s-3} + 2 = \frac{2s-5}{(s-2)(s-3)}$

$\frac{2s-5}{(s-2)(s-3)} = \frac{a}{s-2} + \frac{B}{s-3}$

$a(s-3) + B(s-2) = 2s-5$

$s=2 \quad -a = -1, \quad \boxed{a=1}$

$s=3 \quad \boxed{B=1}$

$y = ae^{2t} + Be^{3t} = \underline{e^{2t} + e^{3t}}, \qquad y(0) = 2,$

3. $\mathcal{L}^{-1}\left(e^{-3s}\left(\frac{s^2+1}{4s^2-1}\right)\right)$ $\quad \frac{1}{4}\mathcal{L}^{-1}\left(e^{-3s}+\frac{5/4}{s^2-\frac{1}{4}}\right)$

$$\mathcal{L}^{-1}\left(\frac{s^2+1}{4s^2-1}\right)=\mathcal{L}^{-1}\left(\frac{1}{4}\left(1+\frac{5/4}{s^2-\frac{1}{4}}\right)\right) \qquad k=\frac{1}{2}$$

$$=\boxed{\frac{1}{4}\left(\delta(t-3)+\frac{5}{2}\mathcal{U}(t-3)\sinh\left(\frac{t-3}{2}\right)\right)}$$

We don need $\mathcal{U}(t-3)$

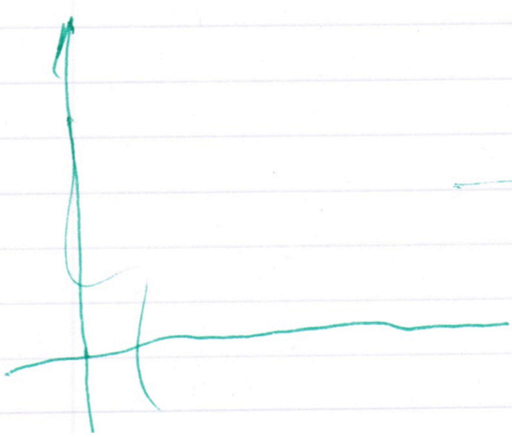
4. $y''+y=\delta(t-\pi),\ y(0)=1,\ y'(0)=0$

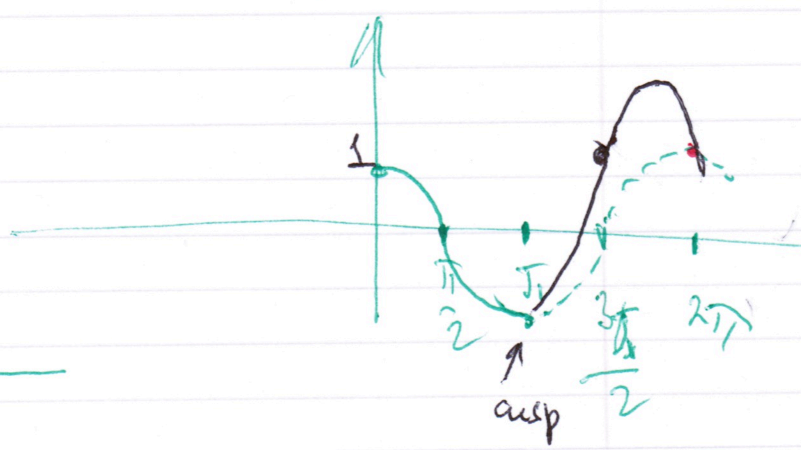
$$(s^2+1)Y=e^{-s\pi}+sy(0)+y'(0)$$

$$(s^2+1)Y=e^{-s\pi}+s$$

$$Y=\frac{e^{-s\pi}+s}{s^2+1}$$

$$y=\cos t+\mathcal{U}(t-\pi)\sin(t-\pi)=\cos t-\mathcal{U}(t-\pi)\sin t.$$





Systems of differential equations

(Sect. 4.6)

$$\begin{cases} x' = x+y & x(0)=1 \\ y' = x-y & y(0)=-1 \end{cases}$$

$x(t), y(t)$

$\rightsquigarrow \quad X = \mathcal{L}x, \ Y = \mathcal{L}y$

$$\begin{cases} sX - x(0) = X+Y \\ sY - y(0) = X - Y \end{cases}$$

$$\begin{cases} (s-1)X - Y = 1 \\ -X + (s+1)Y = -1 \end{cases}$$

$$(s^2-1)\cancel{X} - X = s+1-1$$

$$(s^2-2)X = s$$

$$X = \frac{s}{s^2-2}, \qquad x = \mathcal{L}^{-1}(X) = \boxed{\cosh(\sqrt{2}t)}$$

$(\sinh)'(t) = \cosh t$

$(\cosh)'(t) = \sinh t$

$$y = x' - x = \boxed{\sqrt{2}\sinh(\sqrt{2}t) - \cosh(\sqrt{2}t)}$$

$x(0)=1, \ y(0)=-1$

Another solution.

$$X = \frac{s}{s^2-2} = \frac{a}{s-\sqrt{2}} + \frac{B}{s+\sqrt{2}} = \frac{1}{2}\left(\frac{1}{s-\sqrt{2}} + \frac{1}{s+\sqrt{2}}\right)$$

$$a+B=1 \qquad a=B=\frac{1}{2}$$
$$\sqrt{2}\,(a-B)=0$$

$$x(t) = \frac{1}{2}\left(e^{\sqrt{2}t} + e^{-\sqrt{2}t}\right)$$

$$y(t) = x'(t) - x(t) =$$
$$= \frac{1}{2}\left((\sqrt{2}-1)\,e^{\sqrt{2}t} - (\sqrt{2}+1)\,e^{-\sqrt{2}t}\right).$$