Lecture 6. (Section 4.5)

δ-function (The Dirac Delta function)

$$\delta(t) = \begin{cases} \infty & t=0 \\ 0 & t\neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

It's a "generalized function"

(corresponds to the unit impulse of the duration 0)

Properties:

1) $\int_a^b \delta(t) = \begin{cases} 1 & a<0<b \\ 0 & \text{otherwise} \end{cases}$;

2) $\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$.

3) $\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$

4) $a>0 \quad \mathcal{L}(\delta(t-a)) = \int_0^{\infty} e^{-ts}\delta(t-a)\,dt$

$= e^{-as}$

$$\boxed{\mathcal{L}(\delta(t-a)) = e^{-sa},\ a\geq 0}$$ ⇒ Table 1.

$\mathcal{L}(\delta(t)) = 1$

Now we can compute $\mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}\right)$ if $\deg P = \deg Q$.

Examples.

1. $\mathcal{L}^{-1}\left(\frac{2s^2-4s+9}{s^2-2s+5}\right) =$

$$= \mathcal{L}^{-1}\left(2 - \frac{1}{s^2-2s+5}\right)$$

$$= 2\delta(t) - \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+4}\right)$$

$$= 2\delta(t) - \frac{e^t}{2}\sin 2t.$$

2. $\mathcal{L}^{-1}\left(e^{-2s}\frac{s^2-1}{s^2+4}\right) = \delta(t-2) - \frac{5}{2}u(t-2)\sin(2(t-2))$

$$\mathcal{L}^{-1}\left(\frac{s^2-1}{s^2+4}\right) = \mathcal{L}^{-1}\left(1 - \frac{5}{s^2+4}\right) = \delta(t) - \frac{5}{2}\sin 2t$$

Table 1

Let us to start a table

$f(t)$	$\mathcal{L}f(s)$
t^n, $n=0,1,\dots$	$\frac{n!}{s^{n+1}}$
e^{at}, $a<0$	$\frac{1}{s-a}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$, $s\geq \lvert k\rvert$
$\cosh kt$	$\frac{s}{s^2-k^2}$, $s\geq \lvert k\rvert$
$\delta(t-a)$, $a\geq 0$	e^{-sa}