Lecture 4.

What we know?

1. $\mathcal{L}$ of specific functions functions (Table 1).

2. Table 2.

3. We usualy compute of rational fractions:

$$\mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}\right), \quad \deg P < \deg Q$$

$$Q(s) = \prod_i (s+a_i)^{\alpha_i} \prod_j (s^2+B_j s+C_j)^{\beta_i}$$

Then we apply partial fractions!

We'll not consider problems with $\beta_j > 1$.

Ex. $$\frac{1}{(s^2+4s+13)s^3} = \frac{As+B}{(s^2+4s+13)} + \frac{Cs^2+Ds+E}{s^3}$$

It must be enough of undefined coefficients.

— deg of numerator must be 1 less than the degree of denominator

Derivatives of the Laplace transform

$$\left(\mathcal{L}f\right)'(s) = \left(\int_0^\infty e^{-st} f(t)\,dt\right)' = -\int_0^\infty e^{-st}\, t f(t)\,dt =$$

$$= -\mathcal{L}(t f(t)).$$

$$\mathcal{L}(t f(t)) = -\frac{d}{ds}\mathcal{L}f$$

$\Rightarrow$ Table 2

$$\mathcal{L}(t^n f) = (-1)^n (\mathcal{L}f)^{(n)}(s)$$

[Compare differ. on t and s.

Ex.

1) $\mathcal{L}(t \sin t) = -\left(\frac{1}{s^2+1}\right)' = \frac{2s}{(s^2+1)^2}$

2) $\mathcal{L}(e^{at} t^k) = \frac{k!}{(s-a)^{k+1}}$ (2 ways!)

$\frac{k!}{s^{k+1}} \Rightarrow$

3) $\mathcal{L}(t e^{-3t} \sin 2t) = \boxed{+\frac{4s+12}{(s^2+6s+13)^2}}$

$\mathcal{L}(\sin 2t) = \frac{2}{s^2+4}$

$\mathcal{L}(e^{-3t} \sin 2t) = \frac{2}{(s+3)^2+4} = \frac{2}{s^2+6s+13}$

Transforms of integrals.

The convolution

$$\boxed{f*g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.}$$

It's a "2nd product" of functions

$f*g = g*f$ (commutative)

$(f*g)*h = f*(g*h)$ (associative)

Convolution Theorem

$$\boxed{\mathcal{L}(f*g) = \mathcal{L}f \cdot \mathcal{L}g}$$

(without proof)

Applications:

1) Compute the convolution

$$f(t) = \int_0^t \tau \cos(t-\tau)\,d\tau = 1-\cos t$$

$$\mathcal{L}f = \mathcal{L}(\tau)\,\mathcal{L}(\cos\tau) = \frac{1}{s^2}\cdot\frac{s}{s^2+1} = \frac{1}{s(s^2+1)} =$$

$$= \frac{a}{s} + \frac{Bs+C}{s^2+1}$$

$$\begin{array}{l|l} a+B=0 & a=1,\ B=-1,\ C=0 \\ C=0 & \\ a=1 & \end{array}$$

$$\mathcal{L}f = \frac{1}{s} - \frac{s}{s^2+1}.$$

We need to return back.

$$f = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) = 1 - \cos t.$$

2. Compute

$$f(t) = \int_0^t e^{\tau} \cos(t-\tau)\, d\tau = \frac{1}{2}\left(e^t - \cos t + \sin t\right)$$

$$F = \mathcal{L}f = \frac{1}{s-1} \cdot \frac{s}{s^2+1} = \frac{s}{(s-1)(s^2+1)} = \frac{a}{s-1} + \frac{Bs+C}{s^2+1}$$

$$a(s^2+1) + (s-1)(Bs+C) = s$$

$$\begin{array}{c|l} s^2 & a+B=0 \\ s & -B+C=1 \\ 1 & a-C=0 \end{array} \qquad \begin{array}{l} 2a=1 \\ a=\frac{1}{2},\ B=-\frac{1}{2},\ C=\frac{1}{2} \end{array}$$

$$\mathcal{L}^{-1}F = ae^t + B\cos t + C\sin t$$

$$= \frac{1}{2}\left(e^t - \cos t + \sin t\right)$$

Special case of $f*g$ if $g \equiv 1$,

$$f*1 = \int_0^t f(\tau)\,d\tau \leftarrow \underline{\text{antiderivative}}$$

Corollary.

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{(\mathcal{L}f)(s)}{s}.$$

What is integral equation?

Example. Volterra integral equations,

$$f(t) = g(t) + \int_0^t h(t-\tau) f(\tau)\,d\tau = g(t) + (h*f)($$

- f - unknown function

(g, h) are known

$F = \mathcal{L}f$, $G = \mathcal{L}g$, $H = \mathcal{L}h$

$$F = G + HF$$

$$F = \frac{G}{1-H}, \quad f = \mathcal{L}^{-1}\left(\frac{G}{1-H}\right).$$

Ex.

$$f(t) = \underbrace{e^{3t}}_{g} + \int_0^{\infty} \underbrace{e^{2(t-\tau)}}_{h} f(\tau)\, d\tau$$

$F = \mathcal{L}f$

$$F = \frac{1}{s-3} + \frac{1}{s-2} F$$

$$F = \frac{\frac{1}{s-3}}{1 - \frac{1}{s-2}} = \frac{s-2}{(s-3)^2} \overset{?}{=} \frac{1}{s-3} + \frac{1}{(s-3)^2}$$

$$\underbrace{f(t) = e^{3t}(1+t).}_{\mathcal{L}^{-1}F}$$

Integrodifferential equations

$$y' + 4\int_0^t y(\tau)\, d\tau = 1, \quad y(0) = 1$$

$Y = \mathcal{L}y$

$$\left(s + \frac{4}{s}\right) Y = \frac{1}{s} + 1$$

$$\left(\frac{s^2+4}{s}\right) Y = \frac{s+1}{s}$$

$$Y = \frac{s+1}{s^2+4}$$

$$\underbrace{y(t)}_{\mathcal{L}^{-1}Y} = \cos 2t + \frac{1}{2} \sin 2t$$

Selected problems from HA1.

Sect. 4.1.

#1 $f(t) = \begin{cases} -1 & t<1 \\ 1 & t \geq 1 \end{cases} = 1 + \begin{cases} -2 & x<1 \\ 0 & x>1 \end{cases}$

$$\mathcal{L}f(s) = \frac{1}{s} - 2\int_0^1 e^{-st}\,dt = \frac{1}{s} + \frac{2}{s}e^{-st}\Big|_0^1 =$$

$$= \frac{1}{s} + \frac{2}{s}(e^{-s}-1) = \frac{1}{s}(2e^{-s}-1).$$

#5. $f(t) = \begin{cases} \sin t & 0<t<\pi \\ 0 & t>\pi \end{cases}$

$$\int_0^{\pi} = \int_0^{\infty} - \int_{\pi}^{\infty}$$

$$\mathcal{L}f = \mathcal{L}(\sin t) - \int_{\pi}^{\infty} \sin t\, e^{-st} =$$

$t = \pi + u$

$$= \frac{1}{s^2+1} + e^{-s\pi}\underbrace{\int_0^{\infty} \sin u\, e^{-su}\,du}_{\mathcal{L}(\sin u)} =$$

$$= \frac{1}{s^2+1} + e^{-s\pi}/s^2+1 = \frac{e^{-s\pi}}{s^2+1}.$$

4.1.

4

$$f(t) = \begin{cases} 2t+1 & t \le 1 \\ 0 & t > 1 \end{cases}$$

$$\mathcal{L}f = \int_0^1 = \int_0^\infty - \int_1^\infty = \mathcal{L}(2t+1) - \int_1^\infty$$

$$= \frac{2}{s^2} + \frac{1}{s} - \int_{\infty}^{\infty}$$

$$\int_1^\infty (2t+1) \cancel{\quad} e^{-ts}\,dt = e^{-s}\left(\int_0^\infty (2u+3)e^{-tu}\,du\right) =$$

$t = 1+u$

$$= e^{-s}\left(2\mathcal{L}(u) + 3\mathcal{L}(1)\right) =$$

$$= e^{-s}\left(\frac{2}{s^2} + \frac{3}{s}\right)$$

$$\mathcal{L}f = \boxed{\frac{1}{s^2}\left(2+s - e^{-s}(2+3s)\right)}$$

$$= \cancel{\frac{e^{-s\pi}}{s^2+1}}$$

[Other solutions.]

#13 $\mathcal{L}(te^{4t}) = \frac{1}{(s-4)^2}$, $s>4$

#23 $f(t) = t^2 + 6t - 3$

$$\mathcal{L}f = \frac{2}{s^3} + \frac{6}{s^2} - \frac{3}{s}$$

#28 $f(t) = t^2 - e^{-9t} + 5$

$$\mathcal{L}f(s) = \frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}$$

#37 $f(t) = \sin 2t \cos 2t = \frac{1}{2} \sin 4t$

$$\mathcal{L}f(s) = \frac{4}{2} \frac{1}{s^2+16} = \frac{2}{s^2+16}$$

#38 $f(t) = \cos^2 t = \frac{1}{2}(\cos 2t + 1)$

$$\mathcal{L}f(s) = \frac{1}{2}\left(\frac{s}{s^2+4} + \frac{1}{s}\right).$$

2.5.

Table 2

$f(t)$	$\mathcal{L}f(s)$, $s>0$
$f^{(k)}(t)$	$s^k \mathcal{L}f(s) - s^{k-1}f(0) - \cdots - f^{(k-1)}(0)$
$e^{at}f(t)$	$\mathcal{L}f(s-a)$, $s > \max(0,a)$
$\mathcal{U}(t-a)\,f(t-a), a>0$	$e^{-sa}\mathcal{L}f$
$\mathcal{U}(t-a)\,g(t), a>0$	$e^{-as}\mathcal{L}g(t+a)$
$t^n f(t)$	$(-1)^n \mathcal{L}^{(n)}f(s)$
$\int_0^t f(\tau)\,d\tau$	$\frac{\mathcal{L}f}{s}$
$f*g$	$\mathcal{L}f \cdot \mathcal{L}g$