Lecture 3.

Translations 1.

Let we know $\mathcal{L}f = F$.

Find $\mathcal{L}(e^{at}f(t))$.

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\,dt =$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt = (\mathcal{L}f)(s-a).$$

Sign of a? $\begin{cases} a<0 \text{ - no problems for } s>0 \\ a>0 \text{ - only for } s>a \end{cases}$

$$\mathcal{L}(e^{at}f(t)) = (\mathcal{L}f)(s-a), \quad s > \max(0,a)$$

Ex. 1) $\mathcal{L}(e^{3t}\cos 5t) = \dfrac{s-3}{(s-3)^2+25} = \dfrac{s-3}{s^2-6s+34}$.

$\mathcal{L}(\cos 5t) = \dfrac{s}{s^2+25}$

Applications to $\mathcal{L}^{-1}$

2) $\mathcal{L}^{-1}\left(\frac{5}{(s-5)^3}\right) = \frac{5}{2} t^2 e^{5t}$, $s > 5$

$3(s+1)-4$

3) $\mathcal{L}^{-1}\left(\frac{3s-1}{(s+1)^2}\right) = \mathcal{L}^{-1}\left(\frac{3}{s+1} - \frac{4}{(s+1)^2}\right) =$

$\tilde{s} = s+1$

$= 3e^{-t} - 4e^{-t}t = e^{-t}(3-4t)$.

4) $\mathcal{L}^{-1}\left(\frac{2s+3}{s^2+6s+13}\right) = \mathcal{L}^{-1}\left(\frac{2(s+3)-3}{(s+3)^2+4}\right)$

$\tilde{s} = s+3$

No real roots   Start of denominator

$= e^{-3t}\left(2\cos 2t - \frac{3}{2}\sin 2t\right)$.

corresponds to the transl.   - ~~[illegible]~~ start from denominator!

5) $\mathcal{L}^{-1}\left(\frac{3s-1}{s^2+4s+1}\right) = \mathcal{L}^{-1}\left(\frac{3(s+2)-7}{(s+2)^2-3}\right) =$

Roots! $2 \pm \sqrt{3}$

$= e^{-2t}\left(3\cosh(\sqrt{3}t) - \frac{7}{\sqrt{3}}\sinh(\sqrt{3}t)\right)$.

Another solution.

$\mathcal{L}^{-1}\left(\frac{3s-1}{(s+2-\sqrt{3})(s+2+\sqrt{3})}\right) = \mathcal{L}^{-1}\left(\frac{A}{s+2-\sqrt{3}} + \frac{B}{s+2+\sqrt{3}}\right)$

$$\begin{cases} a+B=3 \\ a(2+\sqrt{3})+B(2-\sqrt{3}) = 2(a+B)+\sqrt{3}(a-B) = -1 \end{cases}$$

$$\sqrt{3}(a-B)-7=0 \Rightarrow a-B = \frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

$$a = \frac{9+7\sqrt{3}}{6}, \quad B = \frac{9-7\sqrt{3}}{6}$$

$$\mathcal{L}^{-1}\left(\quad\quad\quad \right) = a e^{-2+\sqrt{3}} + B e^{-2-\sqrt{3}}$$

We have 2 different answers!

Are they different?

No, Use formulas for cosh, sinh.

Conclusion: If $\Delta > 0$ we can write the answer either through hyperbolic or through exponents

Example. Solve the d.e.

$$y'' + 2y' + 2y = e^{2t}, \quad y(0) = 1, \quad y'(0) = 1.$$

$Y = \mathcal{L}y$

$$(s^2 + \underbrace{2s + 2)}_{(s+1)^2+1} Y = \frac{1}{s-2} + sy(0) + y'(0) + 2y(0)$$

$$= \frac{1}{s-2} - s - 1 = \frac{-s^2+s+3}{s-2}$$

$$Y = \frac{-s^2+s+3}{(s-2)((s+1)^2+1)} = \frac{a}{s-2} + \frac{B + C(s+1)}{(s+1)^2+1},$$

$\tilde{s} = s+1$

$$y = a e^{2t} + e^{-t}(B \sin t + C \cos t)$$

$$a(s^2+2s+2) + B(s-2) + C(s+1)(s-2) = -s^2+s+3$$

s^2	$a + C = -1$	$C = -a-1$
s	$2a + B - C = 1$	$3a + B = 0$
1	$2a - 2B - 2C = 3$	$4a - 2B = 1 \quad 10a = 1$

$$a = \frac{1}{10}, \quad B = -\frac{3}{10}, \quad C = -\frac{11}{10}$$

$$y(t) = \frac{1}{10} e^{2t} + e^{-t}\left(-\frac{3}{10}\sin t - \frac{11}{10}\cos t\right)$$

$y(0) = 1$

Translation 2.

The unit step function $U(t)$. (The Heviside function)

$$U(t)=\begin{cases}1 & t>0\\ 0 & t\le 0.\end{cases}$$

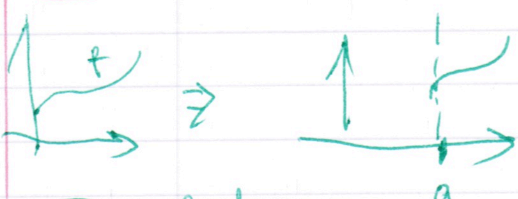
$$U(t-a)=\begin{cases}1 & t\geq a\\ 0 & t\le a\end{cases}$$

2nd Transation Theorem.

$$\mathcal{L}(\underbrace{U(t-a)f(t-a)}_{\text{What is it?}})=\int_0^\infty U(t-a)\,f(t-a)\,e^{-st}\,dt=$$

$\boxed{a>0}$

$$=\int_a^\infty f(t-a)\,e^{-st}\,dt$$



$\boxed{u=t-a}$ $t=u+a$

Translation and cutting.

$$=\int_0^\infty f(u)\,e^{-s(u+a)}\,du=e^{-sa}(\mathcal{L}f)(s)$$

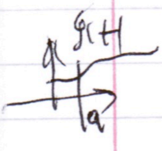
$$\boxed{\mathcal{L}(U(t-a)f(t-a))=e^{-sa}\mathcal{L}f,\ a\geq 0}$$

3rd line at Table 2.

Ex. 1. $\mathcal{L}(U(t-1)(t-1)^2)=e^{-s}\cdot\frac{2}{s^3}$

2. $\mathcal{L}^{-1}\left(\frac{s\,e^{-2s}}{s^2+9}\right)=U(t-2)\cos(3(t-2))$.

Another form of 2nd Translation Theorem.



$$\boxed{\mathcal{L}(g(t)\,\mathcal{U}(t-a)) = e^{-as}\mathcal{L}g(t+a), \quad a>0}$$

$f(t)=g(t+a) \qquad g(t)=f(t-a)$

Examples.

1. $\mathcal{L}(t^2\,\mathcal{U}(t-2)) = e^{-2s}\mathcal{L}((t+2)^2) = e^{-2s}\mathcal{L}(t^2+4t+4)$

$a=2$

$$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$$

2. $\mathcal{L}(\sin t\,\mathcal{U}(t-\frac{\pi}{2})) = e^{-\pi/2\,s}\mathcal{L}\left(\sin\left(\frac{\pi}{2}+t\right)\right) =$

$g(t)=\sin t$
$a=\pi/2$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}(\cos t) = e^{-\frac{\pi}{2}s}\,\frac{s}{s^2+1}$$

3. $\mathcal{L}^{-1}\left(\frac{e^{-3s}}{(s+1)^3}\right) = \frac{1}{2}\mathcal{U}(t-3)\,e^{-t+3}(t-3)^2.$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^3}\right) = \frac{1}{2}e^{-t}t^2$$

4. $\mathcal{L}^{-1}\left(\frac{e^{-5s}}{s^2+9}\right) = \frac{1}{3}\mathcal{U}(t-5)\sin 3(t-5).$

Selected problems from HQ 1

Sect. 4.1.

1) $f(t) = \begin{cases} -1 & t < 1 \\ 1 & t \geq 1 \end{cases} = 1 + \begin{cases} -2 & x < 1 \\ 0 & x > 1 \end{cases}$