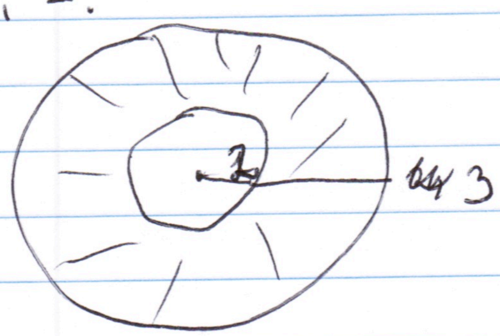
Lecture 26.

Applications of polar coordinates.

Ex. 1.



$\Delta u = 0$

(1) $\Delta u = 0$, $2 < r < 3$, $-\pi < \theta < \pi$

(2) $u_r(2, \theta) = \theta$

(3) $u(3, \theta) = |\theta|$

$$u_n(r,\theta) = R_n(r)\,\Theta_n(\theta)$$

(4) $$u(r,\theta) = c_0 + c_1 \ln r + \sum_{n=1}^{\infty} \{ r^n (a_n \cos n\theta + b_n \sin n\theta) + r^{-n}(d_n \cos n\theta + e_n \sin n\theta) \}$$

$$R_n(r) = c_n r^n + c_{-n} r^{-n}, \quad n > 0$$
$$= c_0 + c_1 \ln r, \quad n = 0$$

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad n = 0, 1, \ldots$$

We minimize the number of undefinite coefficients at

$$u(r,\theta) = \sum u_n(r,\theta)$$

Computation of coefficients,

$$u(2,\theta) = c_0 + c_1 \ln 2 + \sum_{n=1}^{\infty} \{ (2^n a_n + 2^{-n} d_n) \cos n\theta$$

$$+ (2^n b_n + 2^{-n} e_n) \sin n\theta \}$$

$$= \theta = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta \quad , -\pi < \theta < \pi$$

$c_0 + c_1 \ln 2 = 0$

$2^n a_n + 2^{-n} d_n = 0$

$2^n b_n + 2^{-n} e_n = \frac{2(-1)^{n+1}}{n}$

$$\cancel{u(3,\theta) = \frac{\pi}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n}}$$

$$u(3,\theta) = c_0 + c_1 \ln 3 + \sum_{n=1}^{\infty} \{ (3^n a_n + 3^{-n} d_n) \cos n\theta$$

$$+ (3^n b_n + 3^{-n} e_n) \sin n\theta \}$$

$$= |\theta| = \frac{\pi}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos n\theta$$

$c_0 + c_1 \ln 3 = \frac{\pi}{2}$

$3^n a_n + 3^{-n} d_n = \frac{1}{\pi n^2} \left((-1)^n - 1 \right)$

$3^n b_n + 3^{-n} e_n = 0$

$$c_0 + c_1 \ln 2 = 0$$

$$c_0 + c_1 \ln 3 = \frac{\pi}{2}$$

$c_0 = -c_1 \ln 2$

$c_0 (\ln 3/2) = \frac{\pi}{2}$

$$\boxed{c_0 = \frac{\pi}{2 \ln(3/2)}}$$

$$\boxed{c_1 = -\frac{\pi}{2 \ln(3/2) \ln 2}}$$

$$\begin{cases} 2^n a_n + 2^{-n} d_n = 0 \\ 3^n a_n + 3^{-n} d_n = \frac{1}{\pi} \frac{(-1)^n - 1}{n^2} \end{cases}$$

$d_n = -4^n a_n$

$$a_n \left(3^n \cancel{4_n} - \frac{4^n}{3^n}\right) = \frac{1}{\pi} \frac{(-1)^n - 1}{n^2}$$

$$a_n = \frac{1}{\pi} \frac{\left((-1)^n - 1\right) 3^n}{n^2 (9^n - 4^n)}, \quad d_n = -\frac{1}{\pi} \frac{\left((-1)^n - 1\right) 12^n}{n^2 (9^n - 4^n)}$$

$$3^n b_n + 3^{-n} e_n = 0$$

$$2^n b_n + 2^{-n} e_n = \frac{2(-1)^{n+1}}{n}$$

$e_n = -9^n b_n$

$$b_n \left(2^n - \frac{9^n}{2^n}\right) = \frac{2(-1)^{n+1}}{n}$$

$$b_n = 2 \frac{(-1)^{n+1} 2^n}{n (4^n - 9^n)}, \quad e_n = 2 \frac{(-1)^n 18^n}{n (4^n - 9^n)}$$

Substitute at (4).

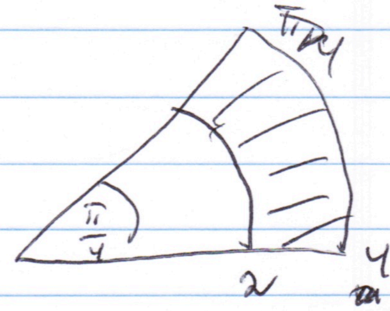
$$u(r,\theta) = \frac{\pi}{2\ln(3/2)}\left(1 - \frac{\ln r}{\ln 2}\right) +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{\pi} \cos n\theta \left(\frac{((-1)^n - 1)\, 3^n}{n^2 (9^n - 4^n)} \left(r^n - 4^n r^{-n}\right)\right)$$

$$+ 2 \sin n\theta \left(\frac{(-1)^{n+1}\, 2^n}{n(4^n - 9^n)} \left(r^n - 9^n r^{-n}\right) \right).$$

$2 < r < 3, \; -\pi < \theta < \pi$

Ex 2.



(1) $\Delta u = 0$, $2<r<4$, $0<\theta<\frac{\pi}{4}$

(2) $u'_\theta(r,0) = u'_\theta(r,\frac{\pi}{4}) = 0$

(3) $u(2,\theta) = 1$, ~~$u(4,\theta)=\theta$~~

(4) $u(4,\theta) = \theta$. $\qquad u_n = R_n \Theta_n$

$\Theta'' + \lambda\Theta = 0$

$\Theta'(0) = \Theta'(\frac{\pi}{4}) = 0$

$\Theta_n(\theta) = \cos(4n\theta)$, $n = 0, 1, \dots$, $\lambda_n = 16n^2$.

$r^2 R'' + rR' - 16n^2 R = 0$

$R_n(r) = c_n r^{4n} + d_n r^{-4n}$, $n > 1$

$c_0 + d_0 \ln r$, $n = 0$

(5) $u(r,\theta) = c_0 + d_0 \ln r + \sum_{n=1}^{\infty} \cos(4n\theta)\left(c_n r^{4n} + d_n r^{-4n}\right)$,

Satisf. (1), (2).

$u(2,\theta) = 1$

$c_0 + d_0 \ln 2 = 1$, $\quad c_0 = (1 - d_0 \ln 2)$

$c_n 2^{4n} + d_n 2^{-4n} = 0$, $\quad d_n = -c_n 2^{8n}$,

$$u(r,\theta) = 1 + d_0 \ln\left(\frac{r}{2}\right) + \sum_{n=1}^{\infty} \cos(4n\theta) \times$$

$$\times\, c_n \left(r^{4n} - \left(\frac{4}{r}\right)^{4n}\right);$$

satisf. (1), (2), (3).

$$u(4,\theta) = 1 + d_0 \ln 2 + \sum_{n=1}^{\infty} \cos(4n\theta)\left(4^{4n} - 1\right) c_n$$

$$= \theta = \frac{\pi}{8} - \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(4n\theta).$$

$$1 + d_0 \ln 2 = \frac{\pi}{8}, \qquad d_0 = \frac{\pi - 8}{8 \ln 2}$$

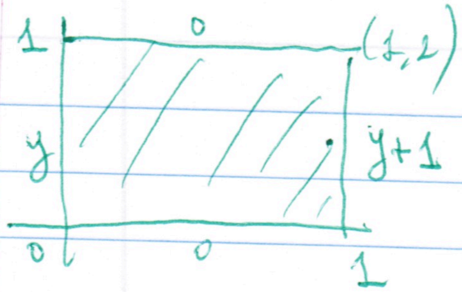
$$c_n\left(4^{4n} - 1\right) = -\frac{1}{2\pi} \frac{(-1)^{-n} - 1}{n^2}, \qquad c_n = -\frac{1}{2\pi} \frac{\left((-1)^{-n} - 1\right)}{\left(4^{4n} - 1\right) n^2}$$

$$u(r,\theta) = 1 + \frac{\pi - 8}{8 \ln 2} \ln \frac{r}{2} + \left(-\frac{1}{2\pi}\right) \sum_{n=1}^{\infty} \cos(4n\theta) \frac{\left((-1)^{-n} - 1\right)}{\left(4^{4n} - 1\right) n^2} \times$$

$$\times \left(r^{4n} - \left(\frac{4}{r}\right)^{4n}\right),$$

$$0 < \theta < \frac{\pi}{4},\ 2 < r < 4$$

26.7



$$\Delta u = u'_{xx} + u'_{yy} = 0 \quad (1)$$

$$u'_y(x,0) = u'_y(x,1) = 0 \quad (2)$$

$$u(0,y) = y \quad (3)$$

$$u(1,y) = y+1 \quad (4)$$

$$u(x,y) = XY$$

$$\frac{Y''}{Y} = -\frac{X''}{X} = -$$

$$Y'' + \lambda Y = 0, \quad 0<y<1$$

$$X'' - \lambda X = 0$$

$$Y'' + \lambda Y = 0$$

$$Y'(0) = Y'(1) = 0$$

$$Y_n = \cos n\pi y, \quad n = 0,1,2,\dots$$

$$\lambda_n = n^2\pi^2$$

$$X_n = a_n \cosh n\pi x + b_n \sinh n\pi x, \quad n>0$$

$$X_0 = a_0 + b_0 x$$

$$u(x,y) = a_0 + b_0 x + \sum_{n=1}^{\infty} \cancel{a_n} \cos(n\pi y)\left(a_n \cosh n\pi x + b_n \sinh n\pi x\right)$$

$$u(0,y) = a_0 + \sum a_n \cos(n\pi y)$$

$$= y = \frac{1}{2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(n\pi y)$$

$$a_0 = \frac{1}{2}, \quad a_n = -\frac{2}{\pi^2} \frac{(-1)^n - 1}{n^2}$$

$$u(1,y) = \left(\frac{1}{2} + b_0\right) + \sum_{n=1}^{\infty}\left(\cos(n\pi y)\left(-\frac{2}{\pi^2}\frac{(-1)^n-1}{n^2}\cosh(n\pi)\right) + b_n \sinh(n\pi)\right)$$

$$= y + 1 = \frac{3}{2} - \frac{2}{\pi^2}\sum \frac{(-1)^n - 1}{n^2} \cos n\pi y$$

$$b_0 = 1$$

$$-\frac{2}{\pi^2}\frac{(-1)^n-1}{n^2} = -\frac{2}{\pi^2}\frac{(-1)^n-1}{n^2}\cosh n\pi + b_n \sinh n\pi$$

$$b_n = \frac{2}{\pi^2}\frac{(-1)^n-1}{n^2}\frac{(\cosh n\pi - 1)}{\sinh n\pi}.$$

$$u(x,y) = \frac{1}{2} + x + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \cancel{\cos(n\pi y)} \cos(n\pi y) \frac{1-(-1)^n}{n^2} \cosh n\pi x$$

$0<x<1$

$0<x<1$

$$\left(\cosh(n\pi x) + \frac{\cosh(n\pi) - 1}{\sinh n\pi}\sinh(n\pi x)\right).$$