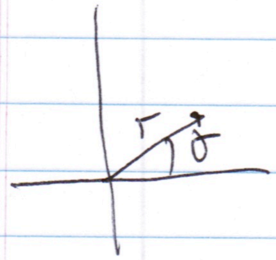
Lecture 25

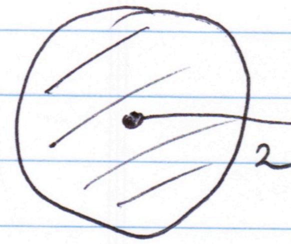
Polar coordinates (Sect. 14.1.)



$u(r,\theta)$

(1) $\Delta u = u''_{rr} + \frac{1}{r} u'_r + \frac{1}{r^2} u''_{\theta\theta} = 0$

$r \le 2$



(2) $u(2,\theta) = \theta, \quad -\pi < \theta < \pi$

Only one boundary condition

Step 1. Separation of variables

$u(r,\theta) = R(r)\Theta(\theta)$

(3) $\begin{cases} \Theta'' + \lambda \Theta = 0 \\ r^2 R'' + rR' - \lambda R = 0 \end{cases}$

$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad n = 0, 1, \dots$

$\lambda = n^2$

$R_n(r) = c_n r^n + d_n r^{-n}, \; n = 1, \dots \; ; \; R_0(r) = c_0 + d_0 \ln r$

$$u(r,\theta)=c_0+d_0\ln r+\sum_{n=1}^{\infty}(c_n r^n+d_n r^{-n})(a_n\cos n\theta+ b_n\sin n\theta).$$

We didn't use yet the boundary condition!

Step 2.

The solution $u(r,\theta)$ is smooth at $r=0$

$$\Downarrow$$

$$d_0=0,\ d_n=0,\ n\geq 1.$$

(4) $$u(r,\theta)=c_0+\sum_{n=1}^{\infty} r^n(a_n\cos(n\theta)+b_n\sin n\theta).$$

$$= \theta, \quad -\pi<\theta<\pi$$

$$u(2,\theta)=\theta=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\theta), \quad \cancel{\theta} \ -\pi<\theta<\pi$$

$$c_0=0,\ a_n=0,\ n\geq 1$$

$$b_n=\frac{(-1)^{n+1}}{n\,2^{n-1}}$$

$$u(r,\theta)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^{n-1}}r^n\sin(n\theta),$$

is the solution of (1), (2). $-\pi<\theta<\pi$, $r\leq 2$

The periodicity and the regularity at the origin replace some boundary conditions!

Example 2. The same problem, but

$\cancel{f}$ $u(\cancel{r}1, \theta) = |\theta|, \quad -\pi < \theta < \pi,$

$$|\theta| = \frac{\pi}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos n\theta .$$

at (4)

$$c_0 = \frac{\pi}{2}, \quad b_n = 0, \quad n \geq 1$$

$$a_n = \frac{1}{\pi} \frac{(-1)^n - 1}{2^n n^2}, \quad n \geq 1$$

$$\underline{u(r, \theta) = \frac{\pi}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty}} \frac{(-1)^n - 1}{2^n n^2} r^n \cos(n\theta)$$

Example 3.

(2) $u(\cancel{0}\ 2, \theta) = \sin 3\theta, \quad -\pi < \theta < \pi$

$$c_0 = 0; \ a_n = 0, n \geq 1, \ b_n = 0, n \neq 3,$$

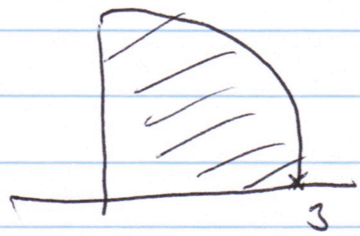
$$u(r, \theta) = \frac{r^3}{\cancel{4}\, 12} \sin 3\theta .$$

Ex. 34.

(3) $\Delta u = 0$, $\quad r \leq 3,\ 0 \leq \theta \leq \frac{\pi}{2}$.

(1) $u(3,\theta) = \theta,\ 0 \leq \theta \leq \frac{\pi}{2}$

(2) $u'_\theta(r,0) = u'_\theta(r,\frac{\pi}{2}) = 0$



$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\Theta'' + \lambda\Theta = 0$$

$$r^2R'' + rR' - \lambda R = 0$$

$$\Theta'(0) = \Theta'(\tfrac{\pi}{2}) = 0$$

$$\Theta_n(\theta) = \cos(2n\theta),\quad 0 \leq \theta \leq \frac{\pi}{2},\quad n = 0, 1, 2, \ldots$$

$$\lambda_n = 4n^2.$$

$\cancel{u(r,\theta)=}$ $R_n(r) = r^{2n},\ n = 1, 2, \ldots$

$R_0(r) = c_0$

Regularity at $r = 0$.

$$u(r,\theta) = c_0 + \sum_{n=1}^{\infty} c_n r^{2n} \cos(2n\theta),\quad r \leq 3,\ 0 < \theta < \frac{\pi}{2}$$

$$u(3,\theta) = c_0 + \sum c_n 3^{2n} \cos(2n\theta) \equiv 0$$

$$\theta = \frac{\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos(2n\theta)$$

Compare coefficients:

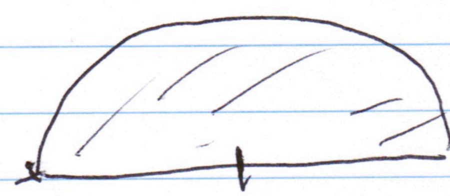
$$c_0 = \frac{\pi}{4}, \quad c_n = \frac{1}{\pi} \, \frac{1-(-1)^n}{3^{4n} n^2}$$

$$u(r,\theta) = \frac{\pi}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \left(\frac{r}{3}\right)^{4n} \cos(4n\theta),$$

$$r \le 3, \; -\pi < \theta < \pi$$

Ex. 5

$$\begin{cases} \Delta u = 0 \\ u(4,\theta) = 2 \\ u(r,0) = u(r,\pi) = 0 \end{cases} \qquad 0 < r < 4, \quad 0 < \theta < \pi$$



$$\Theta'' + \lambda \Theta = 0 \qquad 0 < \theta < \pi$$
$$\Theta(0) = \Theta(\pi) = 0$$

$$\Theta_n(\theta) = \sin(n\theta), \; \lambda_n = n^2, \; n = 1, 2, \ldots$$

$$R_n(r) = r^n, \quad \text{(regularity at } r=0\text{)}.$$

$$u(r,\theta) = \sum_{n=1}^{\infty} c_n r^n \sin(n\theta)$$

$$u(2,\theta) = \sum_{n=1}^{\infty} c_n 2^n \sin(n\theta) = 2$$

$$2 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin n\theta$$

$$c_n = \frac{4}{\pi} \, \frac{1-(-1)^n}{2^n n}$$

$$u(r,\theta) = \cancel{\sum} \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \left(\frac{r}{2}\right)^n \sin(n\theta)$$

$$0<r<2,$$