## MATH 421.03. ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 5

**1.** (75 points) Solve the wave equation

(1) $$9\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, 0 < x < \pi, t > 0$$

subject to the given conditions

(2) $$u(0, t) = u(\pi, t) = 0,$$

(3), (4) $$u(x, 0) = 1, \frac{\partial u}{\partial t}(x, 0) = -1.$$

$$\frac{X''}{X} = \frac{T''}{9T} = -\lambda \; ; \quad X'' + \lambda X = 0, \quad T'' + 9\lambda T = 0$$

$X'' + \lambda X = 0$

$X(0) = X(\pi) = 0$

$X_n = \sin nx$, $n = 1, 2, \dots$

$\lambda_n = n^2$

$T_n = a_n \cos 3nt + b_n \sin 3nt$

$u_n(x,t) = \sin nx\,(a_n \cos 3nt + b_n \sin 3nt)$, $n = 1, 2, 3, \dots$

are solutions of (1) ~~a~~

- with separated variables
- satisfying (2).

$$u(x,t) = \sum_{n=1} u_n(x,t) = \sum_{n=1} \sin nx \left(a_n \cos 3nt + b_n \sin 3nt\right)$$

~~also~~ satisfy (1), (2).

Now the initial conditions (3), (4)

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx \equiv 1, \quad 0 < x < \pi$$

$$1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx \Rightarrow$$

$$a_n = \frac{2}{\pi} \frac{1-(-1)^n}{n}$$

$$u'_t(x,0) = \sum_{n=1}^{\infty} 3n\, b_n \sin nx \equiv -1, \quad b_n = -a_n$$

$$b_n = -\frac{2}{\pi} \frac{1-(-1)^n}{3n^2},$$

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx \left(\cos 3nt - \frac{\sin 3nt}{3n}\right),$$

$0 \leqslant x < \pi$

$t > 0$

**2.** (75 points) Solve the heat equation

$$\pi^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, 0 < x < 2, t > 0$$

subject to the given conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(2,t) = 0,$$

$$u(x,0) = -x.$$

$$X'' + \lambda X = 0$$

$$T' \cancel{\phantom{xx}} + \pi^2 \lambda T = 0 \qquad T_\lambda(t) = \exp(-\pi^2 \lambda t)$$

$$X'(0) = X'(2) = 0$$

$$X_n = \cos\left(\frac{n\pi x}{2}\right), n = 0, 1, 2, \ldots, \quad \lambda_n = \frac{n^2\pi^2}{4}$$

$$T_n = \exp\left(-\frac{n^2\pi^4}{4} t\right)$$

$$u_n(x,t) = X_n T_n; \quad \cancel{u(x,t)}$$

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^4}{4} t\right) \cos\left(\frac{n\pi x}{2}\right),$$

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right), \quad 0 < x < 2$$

$$= -x = -1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos\left(\frac{n\pi}{2} x\right), 0 < x$$

$$u(x,t) = -1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos\left(\frac{n\pi x}{2}\right) \exp\left(-\frac{n^2\pi^4}{4} t\right).$$

$0 < x < 2$, $t$

**3.** (50 points) Using d'Alambert's formula solve the initial problem for the wave equation

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, -\infty < x < \infty, t > 0$$

subject to the given conditions

$$u(\overset{x,0}{\cancel{0,x}}) = x,$$

$$\frac{\partial u}{\partial t}(x,0) = \cos 2x.$$

$$a = 2$$

$$u = \frac{1}{2}\left(x + at + x - at\right) + \frac{1}{4}\int_{x-2t}^{x+2t} \cos 2x \, dx$$

$$= x + \frac{1}{4}\cdot\frac{1}{2}\left(\sin\left(2(x+2t)\right) - \sin\left(2(x-2t)\right)\right) =$$

$$= x + \frac{1}{4}\cos 2x \sin 4t.$$

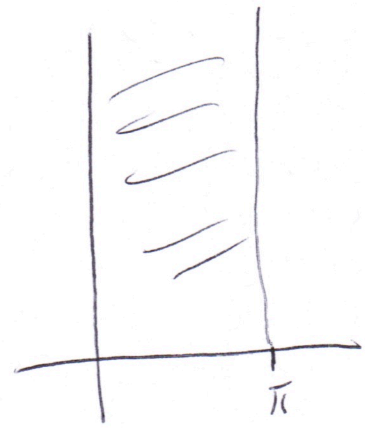
$$u(x,t) = \frac{1}{2}\left(f(x+at) + f(x-at)\right)$$

$$+ \frac{1}{2a}\int_{x-2t}^{x+2t} g(x)\, dx$$

MATH 421.08. ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 5

**1.** (75 points) Solve the wave equation

$$(1) \qquad 9\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, 0 < x < \pi, t > 0$$



subject to the given conditions

$$(2) \qquad u(0,t) = u(\pi,t) = 0,$$

$$(3), (4) \qquad u(x,0) = x+1, \frac{\partial u}{\partial t}(x,0) = x-1.$$

$$\frac{X''}{X} = \frac{T''}{9T} = -\lambda$$

$$X'' + \lambda X = 0$$

$$T'' + 9\lambda T = 0$$

$$X'' + \lambda X = 0 \qquad X_n = \sin nx, \; n = 1, 2, \dots; \; \lambda_n = n^2$$

$$X(0) = X(\pi) = 0 \qquad T_n = a_n \cos 3nt + b_n \sin 3nt.$$

$u_n(x,t) = \sin nx \,(a_n \cos 3nt + b_n \sin 3nt)$, $n = 1, 2, \dots$

are solutions of (1)

- with separated variables

- satisfying (2).

$$u(x,t)=\sum_{n=1}^{\infty} u_n(x,t)=\sum_{n=1}^{\infty} \sin nx \left(a_n \cos(3nt)+b_n \sin(3nt)\right),$$

Satisfy (1), (2).

Conditions (3), (4).

$0<x<\pi$

$$u(x,0)=\sum_{n=1}^{\infty} a_n \sin nx = x+1,$$

$$x=2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

$$1=\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin(nx)$$

$$a_n=\cancel{\frac{2(\pi+1)}{\pi n}(-1)^{n+1}} \cancel{\frac{2(\pi+2)}{\pi n}}=\frac{2}{\pi}\,\frac{(-1)^{n+1}(\pi+1)+1}{n}$$

$$u'_t(x,0)=\sum_{n=1}^{\infty} 3n b_n \sin nx = x-1$$

$$b_n=\frac{2}{\pi}\,\frac{(-1)^{n+1}(\pi-1)-1}{3n^2}.$$

$$u(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cancel{(-1)^{n+1}}\,1}{n} \sin nx \left(\left((\pi+1)+1\right)\cos(3nt)\right.$$

$+$

$$=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}\left(\left((-1)^{n+1}(\pi+1)+1\right)\cos 3nt+\frac{(-1)^{n+1}(\pi-1)-1}{3n}\sin 3nt\right)$$

**2.** (75 points) Solve the heat equation

$$\pi^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, 0 < x < 2, t > 0$$

subject to the given conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(2,t) = 0,$$

$$u(x,0) = 3 - 3x.$$

$$X'' + \lambda X = 0 \qquad T_\lambda(t) = \exp(-\pi^2 \lambda t)$$

$$T' + \pi^2 \lambda T = 0$$

$$X'(0) = X'(2) = 0$$

$$X_n = \cos\left(\frac{n\pi}{2} x\right), n = 0, 1, 2, \dots \; ; \; \lambda_n = \frac{n^2\pi^2}{4},$$

$$T_n = \exp\left(-\frac{n^2\pi^2}{4} t\right), \quad u_n(x,t) = X_n T_n$$

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^4}{4} t\right) \cos\left(\frac{n\pi}{2} x\right), \quad 0 < x < 2.$$

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2} x\right), \quad 0 < x < 2.$$

$$= 3 - 3x$$

$$= +\frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi}{2} x\right); \; a_0 = 0, \; a_n = \frac{12}{\pi^2} \frac{(-1)^n - 1}{n^2}.$$

$$u(x,t) = \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi}{2} x\right) \exp\left(-\frac{n^2\pi^4}{4} t\right), \quad 0 < x < 2, \; t > 0$$

**3.** (50 points) Using d'Alambert's formula solve the initial problem for the wave equation

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, -\infty < x < \infty, t > 0$$

subject to the given conditions

$$u(\overset{x,0}{\cancel{0,t}}) = \sin x,$$

$$\frac{\partial u}{\partial t}(x,0) = x^3.$$

$a = 2$

$$u(x,t) = \frac{1}{2}\left[\sin(x+2t) + \sin(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} \cancel{2t}\ \cancel{t}\ x^3\, dx$$

$$= \sin x \cos 2t + \frac{1}{16}\left[(x+2t)^4 - (x-2t)^4\right]$$

$$= \sin x \cos 2t + 4xt^3_* + x^3 t.$$

$$u(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-2t}^{x+2t} g(x)\, dx.$$