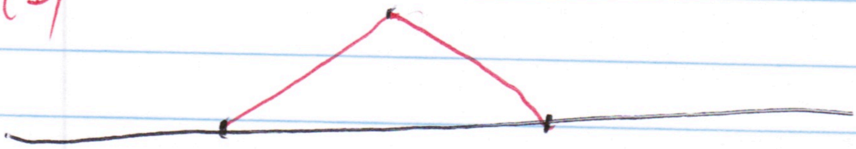
Lecture 23.

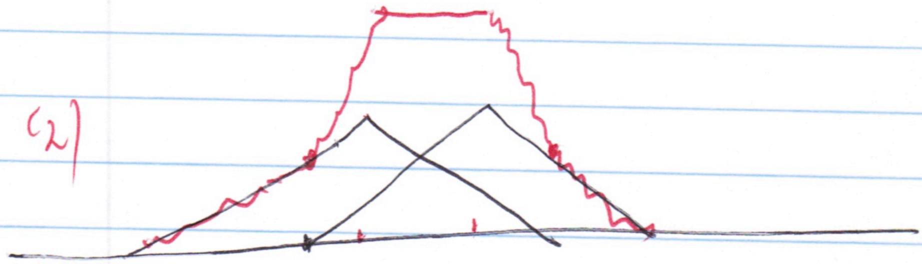
Plucked infinite string.

(1)

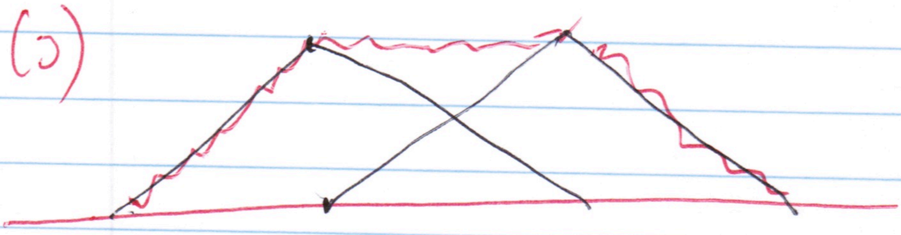


$\frac{1}{2}$ of init. wave

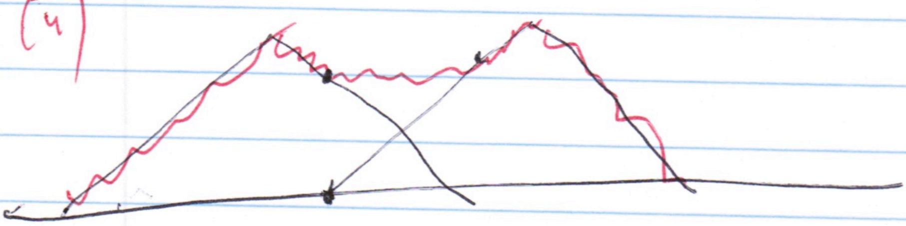
(2)



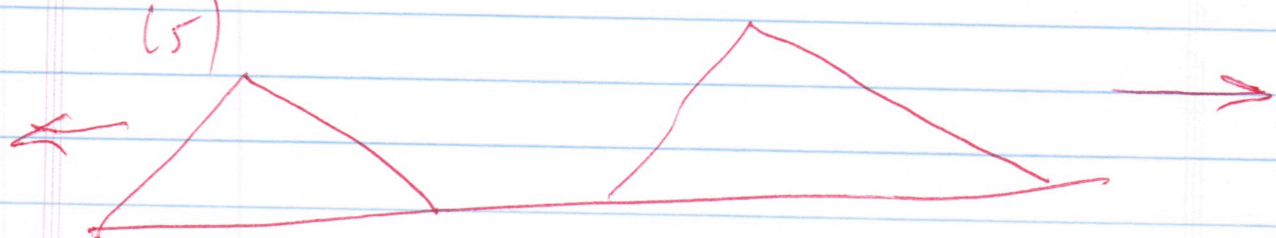
(3)



(4)



(5)



Laplace's Equation (Sect. 13.5)

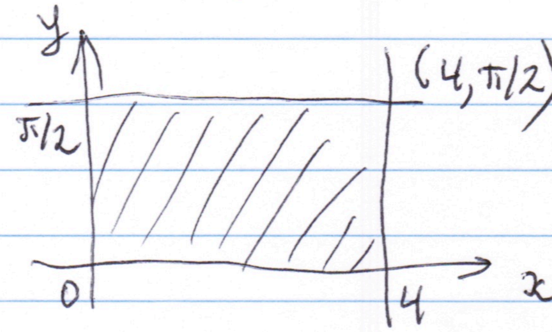
The steady-state temperature of a plate.

(1) $\Delta u = u_{xx} + u_{yy} = 0$;

(2) $u'_x(0,y) + u'_x(4,y) = 0$;

(3) $u(x,0) = 3$;

(4) $u(x, \frac{\pi}{2}) = x$.



Step 1. Separation of variables at (1).

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \qquad u = XY$$

$$(5)\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$$

This system is equivalent to (2) for separated variables.

$$u_\lambda(x,y) = X_\lambda(x)\,Y_\lambda(y).$$

Step 2. Boundary conditions (2).

$$X'' + \lambda X = 0$$
$$X'(0) = X'(4) = 0 \qquad [0,4]$$

$$X_n = \cos\left(\frac{n\pi}{4}x\right), \quad n = 0, 1, 2, \ldots$$
$$\lambda_n = \frac{n^2\pi^2}{16}$$

$$Y_n(y) = a_n \cosh\left(\frac{n\pi}{4}y\right) + b_n \sinh\left(\frac{n\pi}{4}y\right),\ n=1,2,\dots$$

(eigen value $\left(-\frac{n^2\pi^2}{16}\right)$,

$$Y_0(y) = a_0 + b_0 y.$$

$$u(x,y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{4}x\right)\left(a_n \cosh\left(\frac{n\pi}{4}y\right) + b_n \sinh\left(\frac{n\pi}{4}y\right)\right).$$

satisfies (1), (2).

$0 \le x \le 4.$

Step 3. ~~Initial~~ Inhomogeneous conditions (3), (4).

(3) $u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{4}x\right) \equiv 3$,

So $a_0 = 3$, $a_n = 0$, $n > 0$,

(4) $u\left(x, \frac{\pi}{2}\right) = 3 + b_0 \frac{\pi}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{4}x\right) \cdot b_n \sinh\left(\frac{n\pi^2}{8}\right)$

$\equiv x$, $0<x<4$

Cosine Fourier Series:

$$x = 2 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos\left(\frac{n\pi}{4}x\right).$$

$0<x<4$

$3 + b_0 \frac{\pi}{2} = 2; \qquad b_0 \frac{\pi}{2} = -1, \quad b_0 = -\frac{2}{\pi}$

$$b_n \sinh\left(\cancel{4}\, \frac{n\pi^2}{8}\right) = -\frac{8}{\pi^2}\, \frac{1-(-1)^n}{n^2}$$

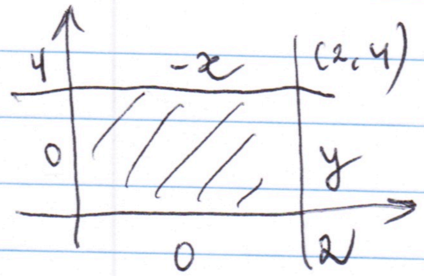
$$b_n = \frac{8}{\pi^2}\, \frac{\left((-1)^n - 1\right)}{n^2 \sinh\left(\frac{n\pi^2}{8}\right)}.$$

$$u(x,y) = 3 \cancel{-} \frac{2}{\pi} y + \cancel{\sum} \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \sinh\left(\frac{n\pi^2}{8}\right)} \times$$

$$\times \cos\left(\frac{n\pi}{4} x\right) \sinh\left(\frac{n\pi}{4} y\right).$$

$0 < x < 4, \; 0 < y < \frac{\pi}{2}$

Superposition principle.



$\Delta u = u_{xx} + u_{yy} = 0$, $0<x<2$, $0<y<4$

$u(0,y)=0$, $u(2,y)=y$

$u(x,0)=0$, $u(x,4)=-x$.

Divide the problem on 2 auxiliary problems

Problem 1.

$$\begin{cases} \Delta v = 0 \\ v(0,y) = v(2,y) = 0 \\ v(x,0) = 0, \ v(x,4) = -x. \end{cases} \qquad 0<x<2,\ 0<y<4.$$

Step 1.

$$v(x,y) = X(x)Y(y)$$

$X'' + \lambda X = 0$

$Y'' - \lambda Y = 0$

~~$X'(0) = X'(2) = 0$~~ $\times$

$X(0) = X(2) = 0$

$X_n = \sin\left(\frac{n\pi}{2}x\right)$, $\lambda_n = \frac{n^2\pi^2}{4}$, $n = 1, 2, \ldots$

$\cancel{\text{Let}}\ Y\cancel{v}_n(x,y) = a_n \cosh\left(\frac{n\pi}{2}y\right) + b_n \sinh\left(\frac{n\pi}{2}y\right)$,

$\cancel{\text{then}}\quad v_n(x,y) = X_n Y_n =$

$$= \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi}{2}x\right)\left(a_n \cosh\left(\frac{n\pi}{2}y\right) + b_n \sinh\left(\frac{n\pi}{2}y\right)\right).$$

$$v(x,y) = \sum_{n=1}^{\infty} v_n(x,y).$$

$v(x,0) = 0 \Rightarrow a_n = 0$ for all n.

$$v(x,4) = \sum_{n=1}^{\infty} b_n \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi}{2}x\right)\sinh(2n\pi\cancel{y}) = -x,\ 0<x<$$

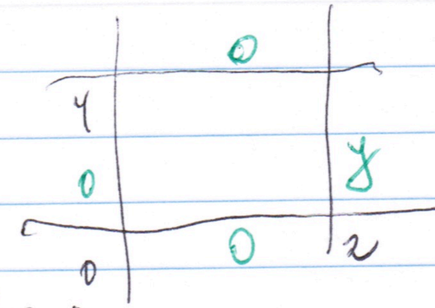
$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\cancel{n\pi}\, n} \sin\left(\frac{n\pi}{2}x\right)$$

$$b_n \sinh(2n\pi) = \frac{4}{\pi}\frac{(-1)^n}{n}$$

$$b_n = \frac{4}{\pi}\frac{(-1)^n}{\sinh(2n\pi)}$$

$$v(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\sinh(2n\pi)}\sin\left(\frac{n\pi}{2}x\right)\sinh\left(\frac{n\pi}{2}y\right)$$

Problem 2.



$\Delta w = 0$

$w(x,0) = w(x,4) = 0$

$w(0,y) = 0, \; w(2,y) = y$

Step 1. $w(x,y) = X(x)\,Y(y)$

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda$$

$$Y'' + \lambda Y = 0 \qquad Y(0) = Y(4) = 0$$

$$X'' - \lambda X = 0$$

$$Y_n(y) = \sin\left(\frac{n\pi}{4} y\right), \quad n = 1, 2, \dots$$

$$X_n = a_n \cosh\left(\frac{n\pi}{4} x\right) + b_n \sinh\left(\frac{n\pi}{4} x\right)$$

$$w_n(x,y) = X_n Y_n$$

$$w(x,y) = \sum_{n=1}^{\infty} \underline{\sin\left(\frac{n\pi}{4} y\right)} \left(a_n \cosh\left(\frac{n\pi}{4} x\right) + b_n \sinh\left(\frac{n\pi}{4} x\right)\right)$$

$w(0,y) \equiv 0 \Rightarrow a_n = 0$ for all n.

$$w(2,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{4} y\right), \quad 0 < y < 4$$

$$\equiv y.$$

$$y = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{4} y\right)$$

$$b_n \sinh\left(\frac{n\pi}{2}\right) = \frac{8}{\pi} \frac{(-1)^{n+1}}{n}$$

$$b_n = \frac{8}{\pi} \frac{(-1)^{n+1}}{n \sin\left(\frac{n\pi}{2}\right)}$$

$$w(x,y) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sin\left(\frac{n\pi}{2}\right)} \sinh\left(\frac{n\pi}{4} x\right) \sin\left(\frac{n\pi}{4} y\right),$$

$$0 < x < 2, \ 0 < y < 4.$$

Superposition:

$$u(x,y) = v(x,y) + w(x,y)$$

is the unique solution of the initial problem