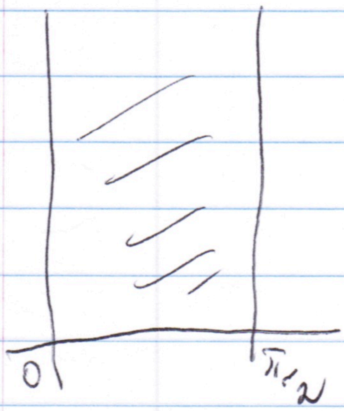
Lecture 22.

1. Exercise (BVP for Heat equation).

equat. (1) $9\frac{\partial^2}{\partial x^2} = \frac{\partial u}{\partial t}$, $0<x<\frac{\pi}{2}$, $t>0$

bound. cond. (2) $u'_x(0,t) = u'_x(\frac{\pi}{2}, t) = 0$, $t>0$

initial cond. (3) $u(x,0) = 2-x$, $0<x<\frac{\pi}{2}$.



a rod with isolated ends.

Step 1. Separation of variables.

$$u(x,t) = X(x)\,T(t)$$

$$\frac{X''}{X} = \frac{T'}{9T} = -\lambda \quad \text{(constant)}.$$

$$(4) \quad \begin{array}{ll} X'' + \lambda X = 0 & 0<x<\frac{\pi}{2} \\ T' + 9\lambda T = 0 & t>0 \end{array}$$

$u_\lambda(x,t) = T_\lambda(t)\, X_\lambda(x)$ is a solution of

$$T_\lambda = \exp(-9\lambda t)$$

Step 2. Boundary conditions (2).

$$(2) \Leftrightarrow X'_\lambda(0) = X'_\lambda\left(\frac{\pi}{2}\right) = 0$$

$$X_n(x) = c_n \cos(2nx), \quad n = 0, 1, 2, \ldots$$

$$\lambda_n = 4n^2.$$

$$u_n(x,t) = c_n \exp(-36 n^2 t) \cos(2nx),$$

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \exp(-36 n^2 t) \cos(2nx).$$

Step 3. The initial condition (3).

$$u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos(2nx) \equiv 2 - x, \quad 0 < x < \frac{\pi}{2}.$$

$$2 - x = \left(2 - \frac{\pi}{4}\right) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(2nx), \quad 0 < x < \frac{\pi}{2}$$

$$c_0 = 2 - \frac{\pi}{4}, \quad c_n = \frac{1}{\pi} \frac{(-1)^n - 1}{n^2}, \quad n > 0.$$

$$c_n = 0, \text{ even } n.$$

$$u(x,t) = 2 - \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{2} \exp(-36 n^2 t) \cos(2nx),$$

$$t > 0, \quad 0 < x < \frac{\pi}{2}.$$

Wave equation Sect. 13.4

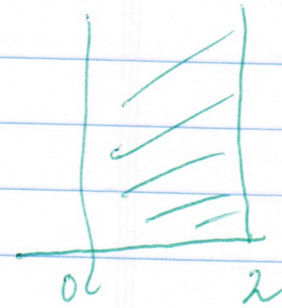
(String's eq.)

(1) $4\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $t>0$, $0<x<2$.

(2) $u(0,t) = u(2,t) = 0, t>0$

(3) $u(x,0) = 1$

(4) $\dfrac{\partial u}{\partial t}(x,t)\Big|_{t=0} = x$



Step 1. Separation o variables

$u(x,t) = X(x)\,T(t)$.

$$\frac{X''}{X} = \frac{T''}{4T} = -\lambda$$

$$(5)\begin{cases} X'' + \lambda X = 0 \\ T'' + 4\lambda T = 0 \end{cases} \qquad 0<x<2,\ t>0$$

$$u_\lambda(x,t) = X_\lambda(x)\,T_\lambda(t)$$

Important: λ is the same at 2 equations.

Step 2. Boundary conditions (2).

$$(2) \Leftrightarrow X_\lambda(0) = X_\lambda(2) = 0$$

(No conditions on $T_\lambda(t)$!).

Eigen functions.

$$X_n(x) = \sin\left(\frac{n\pi x}{2}\right), \quad n = 1, 2, \ldots$$

$$\lambda_n = \frac{n^2\pi^2}{4}.$$

Conclusion:

$$a_n \cos(n\pi t) + b_n \sin(n\pi t)$$

$$u_n(x,t) = \sin\left(\frac{n\pi x}{2}\right)\cancel{\left(a_n \cos\frac{n\pi t}{\ldots} + b_n \sin\frac{n\pi t}{\ldots}\right)}$$

are solutions of (1), (2)

as well as

$$(6) \quad u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\left(a_n \cos(n\pi t) + b_n \sin(n\pi t)\right).$$

We have the freedom in the choice of $\{a_n\}, \{b_n\}$.

Step 3. Initial conditions (3), (4).

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{2}\right), \quad 0 < x < 2.$$

It must be

$$u(x,0) \equiv 1, \quad 0 < x < 2$$

Let us extend 1 at a Sine

Fourier series on $[0,2]$:

$$1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\left(\frac{n\pi x}{2}\right), \quad 0 < x < 2$$

So a_n at (6) are

$$a_n = \frac{2}{\pi} \, \frac{1-(-1)^n}{n}$$

$$u'_t(x,0) = \sum_{n=1}^{\infty} b_n \cdot \frac{n\pi}{2} \cancel{\cos 0} \sin\left(\frac{n\pi x}{2}\right).$$

It must be

$$u'_t(x,0) \equiv x, \quad 0 < x < 2.$$

Let us consider the Sine Fourier

series for $f(x) = x$ on $[0,2]$:

$$x = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{2}\right), \quad 0<x<2.$$

So at (6)

$$b_n \frac{n\pi}{\cancel{}} = \frac{4}{\pi} \frac{(-1)^{n+1}}{n},$$

$$b_n = \frac{4(-1)^{n+1}}{n^2\pi^2}.$$

Finally:

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\left(\frac{1-(-1)^n}{n} \cos\left(n\pi t\right) + \right.$$

$$\left. + \frac{2(-1)^{n+1}}{\pi n^2} \sin\left(n\pi t\right)\right)$$

is the unique solution of (1)-(4).