Part 3. Partial Differential Equations (Ch. 13).

- 2 variables
- Equations of 2-nd order.
- Linear equations

Classical equations:

[Read Sect. 13.2]

(i) $\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}$ - Heat equation

(ii) $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$ - Wave equation (String's equation)

(iii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ - Laplace's equation

Special case of solutions.

Solutions with separated variables

Example. $\frac{\partial^2 u}{\partial x^2} = g \frac{\partial u}{\partial y}$ (1)

Let us seek solutions

$$u(x,y) = X(x)\,Y(y);$$

Substitute at (1):

$$X''Y = g\,XY';$$

Divide on gXY:

$$\frac{X''}{gX} = \frac{Y'}{Y}. \qquad (2)$$

The left part depends of x; the right one of y. So it's a constant. Let $X, Y \neq 0$

$$\frac{X''}{gX} = \frac{Y'}{Y} = -\lambda.$$

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Then

$$X'' + g\lambda X = 0$$
$$Y' + \lambda Y = 0 \qquad (3).$$

(3) is equivalent to (1), if the variables are separated.

<u>All</u> such solutions have a form

$$u_\lambda(x,y) = X_\lambda(x)\, Y_\lambda(y)$$

where X_λ, Y_λ are solutions of (3) for some λ.

$$\boxed{Y_\lambda(y) = c\exp(-\lambda y).}$$

(1) $\lambda > 0$, $\lambda = a^2$, $a > 0$

$$u_a(x,y) = \exp(-a^2 y)\left(c_1 \cos(3ax) + c_2 \sin(3ax)\right);$$

(2) $\lambda < 0$, $\lambda = -b^2$, $b > 0$

$$u_b(x,y) = \exp(b^2 y)\left(d_1 \cosh(3bx) + d_2 \cosh(3bx)\right);$$

(3) $\lambda = 0$

$$u_0(x,y) = f_0 + f_1 x.$$

(independent of y).

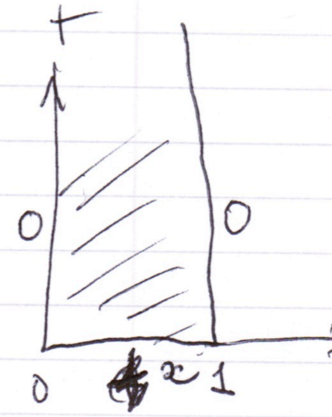
We can take superpositions (linear combinations of these solutions).

BVP for Heat equation (Sect. 13.3)

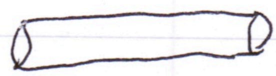
(1) $4\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < 1, \; t > 0$

(2) $u(0,t) = 0, \; u(1,t) = 0, \; t > 0$

(3) $u(x,0) = x, \quad 0 < x < 1$



Step 1. Solutions with separated variables.



Thin rod of length 1; initial temp. is x; the temp. of the ends is zero.

$$u(x,t) = X(x)\,T(t)$$

$$\frac{X''}{X} = \frac{T'}{4T} = -\lambda.$$

We add the eigen value λ artificialy.

$$\left.\begin{aligned} X'' + \lambda X &= 0 \\ T' + 4\lambda T &= 0 \end{aligned}\right\} \quad (4)$$

For separated variables $(4) \Leftrightarrow (1)$.

$$u_\lambda(x,t) = T_\lambda(t)\,X_\lambda(x),$$

$$T_\lambda(t) = \exp(-4\lambda t).$$

We also know possible $X_\lambda(x)$.

Step 2. Boundary conditions (2).

$(2) \Leftrightarrow \quad X_\lambda(0) = X_\lambda(1) = 0.$

$X_\lambda'' + \lambda X_\lambda = 0$

So possible eigen functions

$$X_n(x) = c_n \sin(n\pi x), \quad n = 1, 2, 3, \ldots$$

withe the eigen values $\lambda_n = n^2\pi^2$.

As the result the solutions

$$\boxed{u_n = c_n \exp(-4n^2\pi^2 t) \sin n\pi x}$$

satisfy to the equation (1) and the boundary condition (2). The same for

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-4n^2\pi^2 t) \sin(n\pi x).$$

Step 3. Find $\{c_n\}$ such that $u(x,t)$ satisfies (3) as well and as the result to whole BVP (1-3).

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \equiv x, \quad 0 < x < 1$$

We know that

$$x = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x); \quad 0 < x < 1$$

So $\quad c_n = \frac{2}{\pi} \frac{(-1)^{n+1}}{n}$ and

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \exp(-4n^2\pi^2 t) \frac{(-1)^{n+1}}{n} \sin(n\pi x),$$

is the solution of (1)-(3).

It's possible to show that this solution is unique.

So we found that solutions with seperated variables can be used as a base for the solution of BVP (1)-(3).