Lecture 2.

The inverse Laplace transform $\mathcal{L}^{-1}$.

Definition. $f(t) = \mathcal{L}^{-1}F$ iff $F = \mathcal{L}f$.

Examples. Inverse the table!

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{1}{2}t^2$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2-1}\right) = \cosh t$$

We use by the same table but «from right to left»

$$\mathcal{L}^{-1}(aF + bG) = a\mathcal{L}^{-1}F + b\mathcal{L}^{-1}G$$

Examples.

1. $$\mathcal{L}^{-1}\left(\frac{3s-1}{s^2+4}\right) = 3\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= 3\cos 2t - \frac{1}{2}\sin 2t.$$

⇐ Calculus   Partial Fractions (undefined coefficints)

Rational fractions: $\frac{P(s)}{Q(s)}$

2. $$\mathcal{L}^{-1}\left(\frac{2s}{s^2-3s+2}\right) = \mathcal{L}^{-1}\left(\frac{2s}{(s-2)(s-1)}\right)$$

$$\frac{2s}{(s-2)(s-1)} = \frac{a}{s-2} + \frac{B}{s-1} = \frac{(a+B)s - (a+2B)}{(s-2)(s-1)} =$$

$$\left.\begin{array}{l} a+B=2 \\ a+2B=0 \end{array}\right] \quad a=4,\ B=-2 \qquad = \frac{4}{s-2} - \frac{2}{s-1}.$$

$$= 4e^{2t} - 2e^{t}.$$ (Well defined only for $s>2$)

~~2nd solution~~.

(2) $\mathcal{L}^{-1}\left(\underbrace{\frac{1}{(s+1)(s-2)(s+3)}}_{=}\right) = -\frac{1}{6}e^{-t} + \frac{1}{15}e^{2t} + \frac{1}{10}e^{-3t}$

For $s > 2$

$$\frac{a}{s+1} + \frac{B}{s-2} + \frac{C}{s+3}$$

Numerators

$$a(s-2)(s+3) + B(s+1)(s+3) + C(s+1)(s-2) = 1$$

1st way: Compare coefficients for $s^2, s, 1$, and solve system for a, B, C (3 equations with 3 variables.

2nd way! The substitution.

$s = -1 \Rightarrow -6a = 1 \Rightarrow a = -\frac{1}{6}$

$s = 2 \Rightarrow 15B = 1 \Rightarrow B = \frac{1}{15}$

$s = -3 \Rightarrow 10C = 1 \Rightarrow C = \frac{1}{10}$

$$\left[\begin{array}{l|l} s^2 & a + B + C = 0 \\ s & a + 4B - C = 0 \\ 1 & -6a + 3B - 2 = 1 \end{array}\right.$$

The substitution is impossible Somitimes

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Transforms of Derivatives

$$\underbrace{(\mathcal{L}f')(s)} = \int_0^\infty \overset{u}{e^{-st}}\, \overset{v'}{f'(t)}\,dt =$$

$$= e^{-st}f(t)\Big|_0^\infty + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathcal{L}f}$$

$$= \underbrace{-f(0) + s\mathcal{L}f(s).} \qquad \underbrace{\mathcal{L}f' = -f(0) + s\mathcal{L}f}$$

Generalization (without proof).

$$\mathcal{L}f^{(k)}(s) = s^k \mathcal{L}f(s) - \underbrace{s^{k-1}f(0) - s^{k-2}f'(0) - \dots - f^{(k-1)}(0)}_{\text{boundary terms}}.$$

Let us start Table 2.

For the computation of the Laplace transform of derivative we need to know boundary values at $t=0$.

Applications to ordinary differential equations.

Examples.

1. $y' - 3y = 2, \ y(0) = 2$ (1)

– $y(x)$ satisfies to the equation
– $y(0)=2$ (boundary condition)

Step 1. Take the Laplace transform of both parts of the equation.

$\boxed{Y = \mathcal{L}y}$

$$(s-3)Y - y(0) = \frac{2}{s}.$$

s is the variable. We have ~~one~~ an algebraic equation connecting (Y, s):

$$(s-3)Y = \frac{2}{s} + 2 = \frac{2(s+1)}{s},$$

Step 2.

$$\boxed{Y(s) = \frac{2(s+1)}{s(s-3)}}$$

We solved the algebraic equation!

Step 3. Take ~~the~~ the inverse Laplace transform

$$y(x) = (\mathcal{L}^{-1} Y)(x)$$

Partial fractions.

$$\frac{2(s+1)}{s(s-3)} = 2\left(\frac{a}{s} + \frac{B}{s-3}\right) = \frac{2((a+B)s - 3a)}{s(s-3)}$$

$a + B = 1$

$-3a = \cancel{a}\ 1$

$$\boxed{a = -\frac{1}{3},\ B = \frac{4}{3}}$$

$$y(x) = \mathcal{L}^{-1}\left(\frac{2}{3}\left(-\frac{1}{s} + \frac{4}{s-3}\right)\right) = \frac{2}{3}\left(-1 + 4e^{3x}\right).$$

Verification: $y(0) = 2$

Problem 2.

$$y' + 2y = \cos 3t, \quad y(0) = -1$$

$y(t)$

Step 1. $Y = \mathcal{L} y$

$$(s+2)Y - y(0) = \frac{s}{s^2+9}$$

$$(s+2)Y = \frac{s}{s^2+9} - 1$$

Step 2. $$Y(s) = -\frac{1}{s+2} + \frac{s}{(s^2+9)(s+2)}$$

Step 3. $y(t) = \mathcal{L}^{-1} Y$

$$y(t) = -\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{s}{(s^2+9)(s+2)}\right)$$

$$= -e^{-2t} + \ldots$$

$$\frac{s}{(s^2+9)(s+2)} = \frac{a}{s+2} + \frac{Bs}{s^2+9} + \frac{C}{s^2+9}$$!

$$a(s^2+9) + Bs(s+2) + C(s+2) = s$$

$$\begin{array}{c|l} s^2 & a+B=0 \\ s & 2B+C=1 \\ 1 & 9a+2C=0 \end{array}$$

$B = -a$

$$\left.\begin{array}{l} a = -\frac{2}{13} \\ B = \frac{2}{13} \\ C = \frac{9}{13} \end{array}\right]$$

$$y(t) = (a-1)e^{-2t} + B\cos 3t + \frac{C}{3}\sin 3t.$$

$$= -\frac{15}{13}e^{-2t} + \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t.$$

Substitution! $t=0 \Rightarrow y(0) = -1.$

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Problem 3,

$$y'' - 4y' = e^{-t}, \quad y(0) = 1, \quad y'(0) = -1.$$

$$(s^2 - 4s)Y - sy(0) - y'(0) + 4y(0) = \frac{1}{s+1}$$

$$s(s-4)Y = \frac{1}{s+1} + s - 5$$

$$Y = \frac{1}{(s+1)s(s-4)} + \frac{1}{s-4} - \frac{5}{s(s-4)}$$

$$= \frac{1}{s-4} + \frac{-5s-4}{s(s-4)(s+1)} = \frac{1}{s-4} + \frac{a}{s} + \frac{B}{s-4} + \frac{C}{s+1}$$

$$y(t) = \underline{a + (B+1)e^{4t} + Ce^{-t}}$$

$$a(s-4)(s+1) + Bs(s+1) + Cs(s-4) = -5s - 4$$

$$\begin{array}{l|l|l|l}
s^2 & a + B + C = 0 & a = 1 & a = 1 \\
s & -3a + B - 4C = -5 & B + C = -1 & B = -\frac{6}{5} \\
1 & -4a = -4 & B - 4C = -2 & C = \frac{1}{5}
\end{array}$$

$$y(t) = 1 - \frac{1}{5}e^{4t} + \frac{1}{5}e^{-t}$$

$$y(0) = 1$$

$$y'(0) = -1$$

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Table 2

$f(t)$	$\mathcal{L}f(s)$, $s>0$
$f^{(k)}(t)$	$s^k \mathcal{L}f(s) - s^{k-1}f(0) - \cdots - f^{(k-1)}(0)$
$e^{at}f(t)$	$\mathcal{L}f(s-a)$, $s > \max(0,a)$
$\mathcal{U}(t-a)\,f(t-a), a>0$	$e^{-sa}\,\mathcal{L}f$