MATH 421.03. ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 4

1. (70 points) a)Let the function $f(x) = 1, \pi/2 \leq x \leq \pi$ and $f(x) = 0, 0 \leq x \leq \pi/2$. Expand it on $[0, \pi]$ in the both Fourier cosine and sine series.

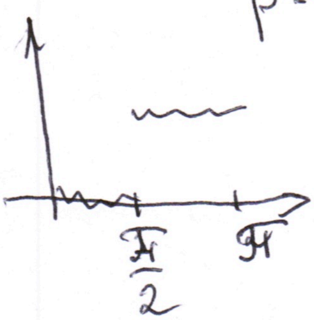
b)Describe the sum of these series for all $-\infty < x < \infty$. Give explanations and sketch the graphs.

c)Specify the numerical series which are the results of evaluations for $x = 0, \pi/2$.

d) Take the sum of these cosine and of this Fouries series and consider it as a Fourier series on $[-\pi, \pi]$. Describe the function which will be its sum.

Cosine

$p = \pi$



$$a_0 = \frac{2}{\pi}\int_{\pi/2}^{\pi} dx = 1$$

$$a_n = \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos(nx)\,dx = \frac{2}{\pi n} \sin nx \Big|_{\frac{\pi}{2}}^{\pi} =$$

$$= \frac{2}{n\pi}\left(\sin \pi n - \sin \frac{n\pi}{2}\right) = \begin{cases} 0, & n = 2k \\ \dfrac{2(-1)^{k+1}}{(2k+1)\pi}, & n = 2k+1 \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} \cos(2k+1)x, \quad 0 < x < \pi$$

Sine

$$b_n = \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin(nx)\,dx = -\frac{2}{n\pi} \cos(nx)\Big|_{\pi/2}^{\pi} =$$

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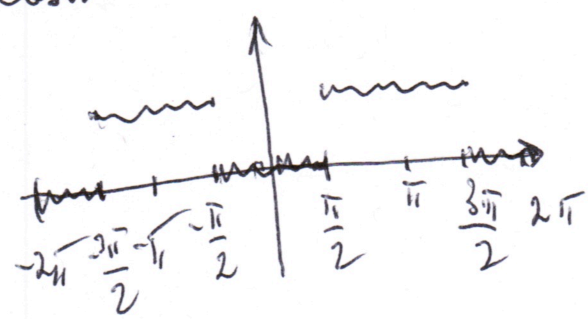
$$= \frac{2}{n\pi}\left((-1)^n - \begin{cases}0, & n=2k+1\\ (-1)^k, & n=2k\end{cases}\right.$$

$$f(x) = \frac{2}{\pi}\left(\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{(2k+1)x}{1}\right) + \right.$$

$$\left. + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}-1}{2l}\sin(2lx)\right), \quad 0<x<\pi$$

b) c)

Cosine



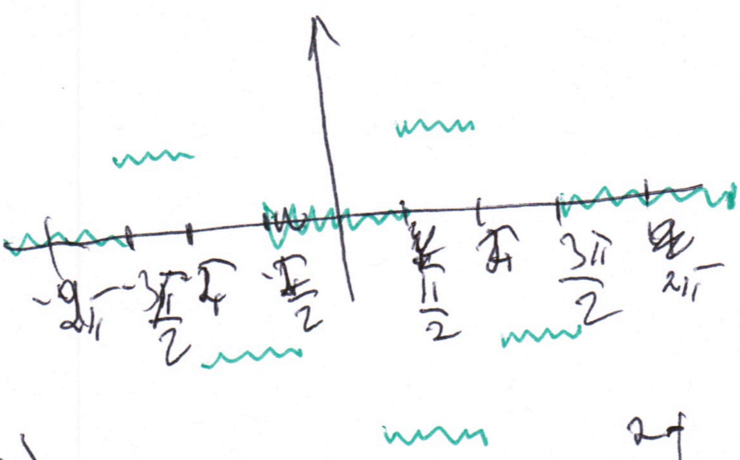
$x=0 \quad 0 = \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$

$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

$x=\frac{\pi}{2} \quad \frac{1}{2}=\frac{1}{2}$

The same

Sine

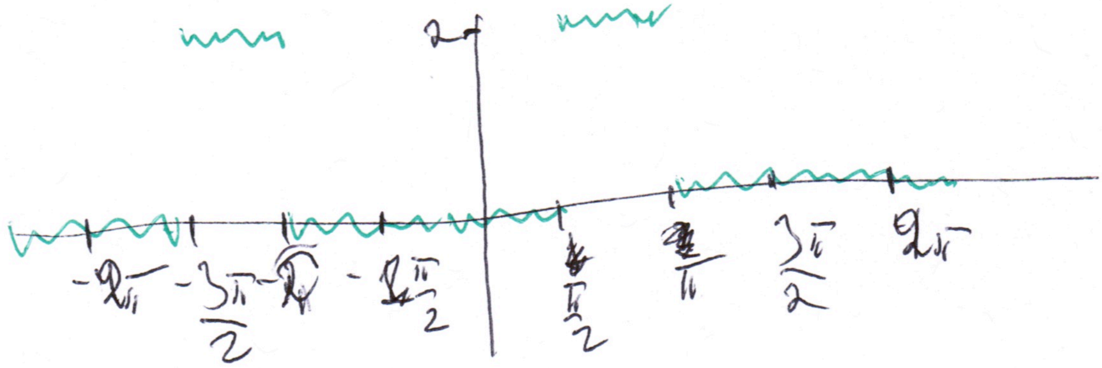


$x=0, \; 0=0$

$x=\frac{\pi}{2}$

$$\frac{1}{2} = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

d)



2. (65 points) a) Give Euler's formula.

b) Expand the function $f(x) = e^{-x}$ in the complex Fourier series on $[-1, 1]$.

c) Describe the sum of this series for all x.

d)Transform this series in the real Fourier series using the formulas connecting coefficients of real and complex series.

a) $e^{ix} = \cos x + i\sin x$

b) $c_n = \frac{1}{2}\int_{-1}^{1} \exp(-x - in\pi x)\,dx = \frac{1}{2}\int_{-1}^{1}\exp(-x(1+in\pi))\,dx$

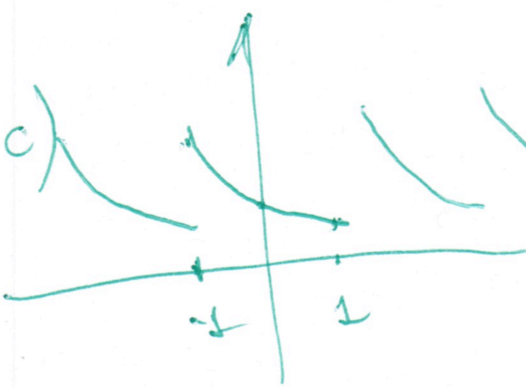
$= -\frac{1}{2}\frac{1}{1+in\pi}\left[\exp(-1-in\pi) - \exp(1+in\pi)\right]$

$e^{in\pi} = (-1)^n$.

$= \frac{(-1)^n}{1+in\pi}\sinh(1).$

$e^{-x} = \sinh(1)\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1+in\pi}\exp(in\pi x), \quad -1<x<1.$

c)



d) $a_n = c_n + c_{-n} = (-1)^n\sinh(1)\left(\frac{1}{1+in\pi} + \frac{1}{1-in\pi}\right) =$

$= \frac{2\sinh(1)}{1+n^2\pi^2}$

$$b_n = i(c_n - c_{-n}) = 2\sinh(1)\,\frac{(-1)^{n+1}\, n\pi}{1+n^2\pi^2}$$

$$e^{-x} = 2\sinh(1)\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2}\Big(\cos(n\pi x) + n\pi\sin(n\pi x)\Big)\right)$$

$-1 < x < 1,$

3. (65 points) Find the eigenvalues and eigenfunctions for the BVP on $[0, \cancel{\pi}1/2]$

$$y'' + \lambda y = 0, y(0) = 0, y(1/2) = 0.$$

1) $\lambda > 0$, $\lambda = a^2$, $a > 0$.

$y_a(x) = c_1 \cos ax + c_2 \sin ax$, $a \neq 0$

$y_a(0) = 0 \Rightarrow c_1 = 0$, $y_a(x) = c \sin ax$

$y_a(1/2) = 0 \Rightarrow \sin \frac{a}{2} = 0 \Rightarrow \frac{a}{2} = n\pi \Rightarrow a = 2n\pi$, $n = 1, 2, \ldots$

$y_n(x) = \sin 2n\pi x$ are eigen ~~values~~ functions with $\lambda_n = 4n^2\pi^2$

2) $\lambda < 0$, $\lambda = -a^2$

$y_a = c_1 \cosh ax + c_2 \sinh ax$

$y_a(0) = 0 \Rightarrow c_1 = 0$, $y_a(x) = c \sinh ax$

$y_a(\frac{1}{2}) = c \sinh \frac{a}{2} = 0 \Rightarrow \cancel{\frac{a}{2}}\ a = 0$ (contradiction)

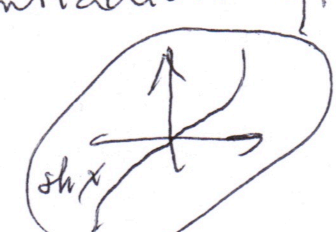
no eigen values with $\lambda < 0$.

3) $\lambda = 0$, $y_0(x) = c_0 + c_1 x$

$y_0(0) = 0 \Rightarrow c_0 = 0$, $y_0 = cx$

$y_0(1/2) = \frac{c}{2} = 0 \Rightarrow c = 0$. Contradiction.

$\lambda = 0$ isn't an eigen value.



Conclusion.

$\sin(2n\pi x)$, $n = 1, 2, \ldots$, are eigen functions

with the eigen values $\lambda_n = 4n^2\pi^2$

4*. (20 extra points). If a function $f(x)$ is even on $[-p, p]$, what can we say about the coefficients of its complex Fourier series? Give a necessary and sufficient condition. Give detailed explanations.

f - even $\Leftrightarrow$ ~~b_n~~ $b_n = 0$

$b_n = i(c_n - c_{-n}) = 0 \Leftrightarrow \boxed{c_{+n} = c_{-n}}$

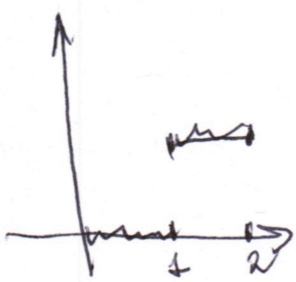
MATH 421.08. ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 4

1. (70 points) a)Let the function $f(x) = 1, 1 \le x \le 2$ and $f(x) = 0, 0 \le x \le 1$. Expand it on $[0, 2]$ in the both Fourier cosine and sine series.

b)Describe the sum of these series for all $-\infty < x < \infty$. Give explanations and sketch the graphs.

c)Specify the numerical series which are the results of evaluations for $x = 0, 1$.

d) Take the sum of these cosine and of this Fouries series and consider it as a Fourier series on $[-2, 2]$. Describe the function which will be its sum.



a) $p=2$ Cosine F. series

$$a_0 = \int_1^2 dx = 1$$

$$a_n = \int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx = \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2 =$$

$$= \frac{2}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) = \begin{cases} 0 & n = 2k \\ \frac{2(-1)^{k+1}}{(2k+1)\pi}, & n = 2k+1 \end{cases}$$

$$\underline{f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2k+1}\cos\frac{(2k+1)\pi}{2}x}, \quad 0<x<2$$

Sine F. series

$$b_n = \int_1^2 \sin\left(\frac{n\pi}{2}x\right)dx = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\Big|_1^2 =$$

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$$= -\frac{2}{n\pi}\left((-1)^n - \begin{cases} 0, & n=2k+1 \\ (-1)^k, & n=2k. \end{cases}\right.$$

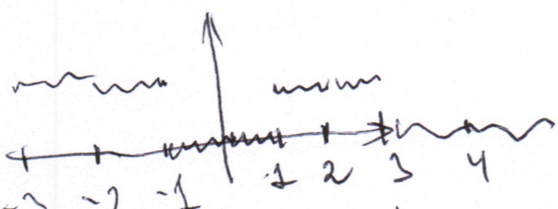
$$f(x) = \frac{2}{\pi}\left(\sum_{k=0}^{\infty} \frac{1}{2k+1}\sin\left(\frac{(2k+1)\pi}{2}x\right) + \right.$$

$$\left. + \sum_{l=1}^{\infty} \underline{\frac{(-1)^{l+1}-1}{2l}} \sin(l\pi x)\right),$$

b), c)

Cosine one

$x=0$; $0 = \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$



$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

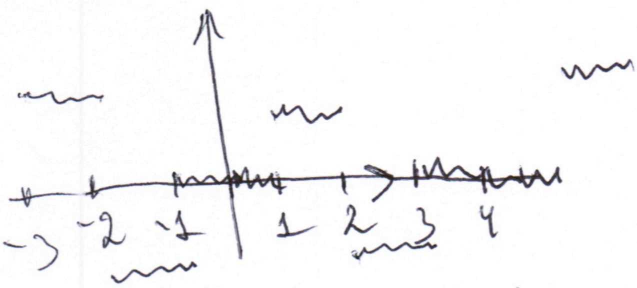
The

su...

$x=1$ $\frac{1}{2} = \frac{1}{2}$

~~Sine on~~

Sine one

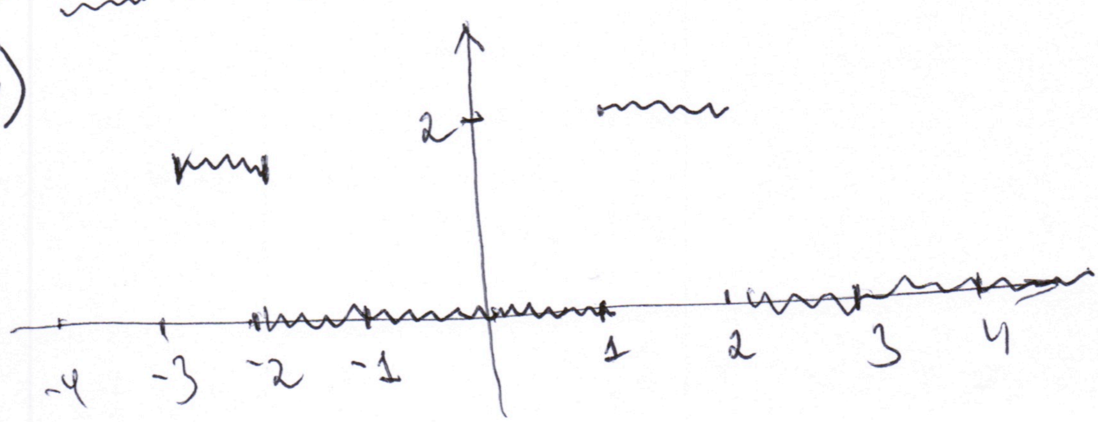


$x=0$, $0=0$

$x=1$

$$\frac{1}{2} = \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

d)



2. (65 points) a) Give Euler's formula.

b) Expand the function $f(x) = e^{\pi x}$ in the complex Fourier series on $[-1, 1]$.

c) Describe the sum of this series for all x.

d) Transform this series in the real Fourier series using the formulas connecting coefficients of real and complex series.

a) $e^{ix} = \cos x + i \sin x$

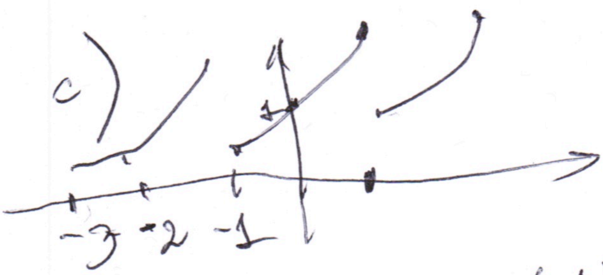
b) $c_n = \frac{1}{2}\int_{-1}^{1} \exp(\pi x - in\pi x)\,dx = \frac{1}{2}\int_{-1}^{1} \exp(\pi x(1-in))\,dx$

$= \frac{1}{2}\,\frac{1}{\pi(1-in)}\left[\exp(\pi(1-in)) - \exp(-\pi(1-in))\right]$

$= \frac{(-1)^n}{\pi(1-in)} \sinh(\pi).$ $\qquad e^{i\pi n} = (-1)^n.$

$e^{\pi x} = \frac{\sinh(\pi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-in} \exp(in\pi x), \; -1<x<1.$

c)



d) $a_n = c_n + c_{-n} = \frac{(-1)^n}{\pi}\sinh(\pi)\left(\frac{1}{1-in} + \frac{1}{in}\right) = \frac{2\sinh\pi\,(-1)^n}{\pi(n^2+1)}$

$b_n = i(c_n - c_{-n}) = \frac{2}{\pi}\sinh(\pi)\frac{(-1)^{n+1}\,n}{n^2+1}$

$e^{\pi x} = \frac{2\sinh(\pi)}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}\left(\cos(n\pi x) + n\sin(n\pi x)\right)\right).$

$-1 < x < 1$

3. (65 points) Find the eigenvalues and eigenfunctions for the BVP on $[0, \pi/2]$

$$y'' + \lambda y = 0, y'(0) = 0, y'(\pi/2) = 0.$$

1) $\lambda > 0$, $\lambda = a^2$, $a > 0$

$y_a(x) = c_1 \cos ax + c_2 \sin ax$

$y_a' = -a c_1 \sin ax + a c_2 \cos ax$, $a \neq 0$

$y_a'(0) = 0 \Rightarrow c_2 = 0$, $y_a(x) = c_1 \cos ax$

$y_a'(\frac{\pi}{2}) = 0 \Rightarrow -a c_1 \sin a\frac{\pi}{2} = 0 \Rightarrow a\frac{\pi}{2} = \pi n \Rightarrow$

$\Rightarrow a = 2n$, $n = 1, \ldots,$

$y_n(x) = \cos 2nx$, $n = 1, 2, \ldots$ are eigen functions with $\lambda_n = 4n^2$;

2) $\lambda < 0$, $\lambda = -a^2$

$y_a(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$

$y_a'(x) = a c_1 \sinh(ax) + a c_2 \cosh(ax)$

$y_a'(0) = 0 \Rightarrow c_2 = 0$, $y_a(x) = \cosh(ax)$

$y_a'(\pi/2) = 0 \Rightarrow \sinh(a\frac{\pi}{2}) = 0$. Impossible if $a \neq 0$.

3) $\lambda = 0$, $y_0(x) = c_0 + c_1 x$

$y_0' = c_1$. $y_0'(0) = 0 \Rightarrow c_1 = 0$, $y_0 = c_0$. Then $y_a' \equiv 0$.

$y_0(x) = 1$ is an eigen function with $\lambda = 0$.

Conclusion!

$y_n(x) = \cancel{\cos(\ldots)}$, $\cos(2nx)$, $n = 0, 1, 2, \ldots$

are eigen functions with $\lambda_n = 4n^2$.

4*. (20 extra points). If a function $f(x)$ is even on $[-p, p]$, what can we say about the coefficients of its complex Fourier series? Give a necessary and sufficient condition. Give detailed explanations.

$b_n = i(c_n - c_{-n}) = 0$ for all n,

So $c_n = c_{-n}$, $n = 0, 1, \ldots$