Lecture 18

Problems from H9.

#9 Laguerre's differential equation

$$xy'' + (1-x)y' + ny = 0$$

$a = x,\quad b = (1-x)$ $\qquad \mu = \exp\left(\int \frac{-x}{x}\,dx\right) = e^{-x}$

$L_n(x)$ - polynomial eigen function orthogonal with the weight e^{-x}.

$r = xe^{-x},\ p = e^{-x}$

$$\boxed{(xe^{-x}y')' + ne^{-x}y = 0}$$

#10. Hermite's differential equation

$$y'' - 2xy' + 2ny = 0$$

$a = 1,\ b = -2x$

$\mu = \exp\left(\int \frac{b}{a}\,dx\right) = \exp\left(\int(-2x)\,dx\right) = \exp(-x^2)$

$H_n(x)$ - polynomial system of eigen functions, orthogonal with the weight e^{-x^2}

$$(e^{-x^2}y')' + 2n\,e^{-x^2}y = 0$$

The important application of St.-L. theory -

compet systems of orthogonal polynomials: (with weights).

Lagerre, Hermite polynomials.

Choise of the interval $[a, b]$

- remember about $r>0, p>0$.

(no restrictions for Laguerre's & Hermite's polynomials.

The most important example

Legendre polynomials (12.6.2), 12.5 (Ex. 4)

Legendre's equation.

$$\underbrace{(1-x^2)y'' - 2xy'}_{b=a'} + n(n+1)y = 0$$

$$((1-x^2)y')' + n(n+1)y = 0$$

$$r = (1-x^2), \ p \equiv 1, \ q \equiv 0$$

Eigen functions are orthogonal (in usual sense, $p \equiv 1$)

Let us consider $[-1, 1]$.

We have $r(x) = (1-x^2) > 0, -1 < x < 1$

$r(-1) = r(1) = 0 \Rightarrow$

Singular St.-L. problem;

no conditions on the ends $x = \pm 1$.

$P_n(x), n = 0, 1, 2, \dots$ — polynomial solutions (Sect. 5.3.2)

$$P_n(x) = c_0 + c_1 x + \dots + c_n x^n$$

Equation $\Rightarrow$ reccurent formulas for c_n.

Normalization: $P(1) = 1$.

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$.

P_{2k} are even, P_{2k+1} — odd.

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0, \quad n \neq m$$

$$\|P_n(x)\|^2 = \int_{-1}^{1} (P_n(x))^2 dx = \frac{2}{2n+1}$$

The Fourier-Legendre Series

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \quad -1 \leq x \leq 1$$

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx.$$

Remember about even and odd functions when compute c_n!