$$y'' + \lambda y = 0 \qquad [a, b]$$

$$\begin{cases} c_1 y(a) + c_2 y'(a) = 0 \\ d_1 y(b) + d_2 y'(b) = 0. \end{cases}$$

Homogeneous boundary conditions

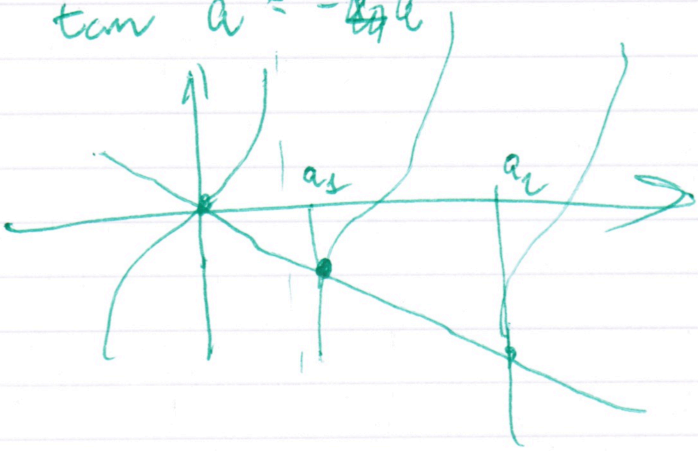
Complet system of eigen values exists always but not always can be computed explicitly.

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad \cancel{y(0)} \; y(1) + y'(1) = 0$$
$[0, 1]$

$\lambda = a^2$, $y_a(0) = 0 \Rightarrow y_a = c \sin ax$, let $c = 1$

$$y_a(1) + y_a'(1) = \sin a + a \cos a = 0 \Rightarrow$$

$$\tan a = -a$$



$\sin a_1 x, \ \sin a_2 x, \ldots$ — eigen functions

with $\lambda_i = a_i^2$ No explicit formula for a_i

Lecture 17.

Sturm-Liouville Problem. Sect. 12.5

Generalization of the equation $y''+\lambda y=0$.

Sturm-Liouville equation

$$\frac{d}{dx}\left(r(x)\,y'\right)+\left(q(x)+\lambda p(x)\right)y=0 \qquad (1)$$

- coefficients $r(x), q(x), p(x)$ (variable) on $[a,b]$
  $r(x)>0$, $p(x)>0$ on $[a,b]$
  (no conditions on $q(x)$).

- Self-adjont form
  $$\frac{d}{dx}\left(r(x)\,y'\right)=r'y'+ry''$$
  $ay''+a'y'$ (discuss later).

$y''+\lambda y=0$, $\qquad r\equiv 1,\ q\equiv 0,\ p\equiv 1$.

Regular Sturm-Liouville Problem

$$\alpha_1 y(a) + B_1 y'(a) = 0 \qquad (2)$$
$$\alpha_2 y(b) + B_2 y'(b) = 0$$

Regularity: $(\alpha_1, B_1), (\alpha_2, B_2)$ are not ~~zeros~~ zero both

(other way there is no conditions!).

Theorem (St.-L.).

(i) There is an infinite number of eigen values
$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots \underset{n\to\infty}{\to} \infty$$

(ii) For each j there is an unique eigen function $y_j(x)$ (up to multiplicative constant).

(iii) Eigen functions $y_j(x)$ for different λ_j are linear independent

(iv) They are orthogonal with the weight $p(x)$:
$$\int_a^b y_i(x) y_j(x) p(x) dx = 0, \; i \neq j$$

Examples:

1) $y'' + \lambda y = 0$

2) <u>Cauchy - Euler equation.</u>

$$x^2 y'' + x y' - \alpha^2 y = 0, \quad \alpha \geq 0$$

$$\lambda = -\alpha^2$$

General solution:

$$y = c_1 x^{-\alpha} + c_2 x^{\alpha}, \quad \alpha \neq 0$$

$$y = c_1 + c_2 \ln x, \quad \alpha = 0.$$

verify

It's not the self-adjoint form.

$$x y'' + y' - \alpha^2 \frac{y}{x} = 0$$

$$(x y')' - \alpha^2 \frac{y}{x} = 0$$

$$\underline{r(x) = x, \; q \equiv 0, \; p} = \frac{1}{x}, \quad \lambda = -\alpha^2$$

How to transform an arbitrary differential equation to the self-adjoint form?

— By multiplying by an integrating factor $\mu(x)$ (positive).

$$\mu(x)\left(a(x)y''+b(x)y'\right) = \left(r(x)y'\right)' = ry''+r'y'$$

Express μ, r through a, b

$\mu a = r$
$\mu b = r'$

$$(\mu a)' = \mu b$$

$$\mu a' + \mu' a = \mu b$$

$$\frac{\mu'}{\mu} = \frac{b-a'}{a}$$

$$\overset{\shortparallel}{(\ln \mu)'}$$

$$\ln \mu = \int \frac{b-a'}{a}\,dx, \quad \mu = \exp\left(\int\left(\frac{b-a'}{a}\right)dx\right).$$

$\mu \neq 0$

$r = a\mu.$

[It's easier to repeat this computation each time]

Important: Verify the sign of r, p after the multiplication on $\mu(x)$.

Examples.

1. Cauchy - Euler eq.

$$x^2 y'' + x y' - \alpha^2 y = 0$$

$$a = x^2, \quad b = x$$

$$\frac{b - a'}{a} = \frac{x - 2x}{x^2} = -\frac{1}{x}$$

$$\mu(x) = \exp\left(-\int \frac{dx}{x}\right) = \exp(-\ln x) = \frac{1}{x}$$

$$\Downarrow$$

$$\cancel{x}\ x y'' + y' - \alpha^2 \frac{y}{x} = 0$$

$$(x y')' - \alpha^2 \frac{y}{x} = 0$$

2. Bessel eq.

$$x y'' + y' + \alpha x y = 0$$

$$(x y')' + \alpha x y = 0$$

$$r = x, \; q = 0, \; p = x, \; \lambda = \alpha$$

3. $x^2 y'' + 3x y' + \lambda y = 0$

Guess: $\mu(x) = x \Rightarrow x^3 y'' + 3x^2 y' + \lambda x y = 0$

$$(x^3 y')' + \lambda x y = 0$$

$$r = x^3, \; q = 0, \; p = x.$$

$$\frac{b - a'}{a} = \frac{3x - 2x}{x^2} = \frac{1}{x}$$

$$\mu(x) = \exp\left(\int \frac{dx}{x}\right) = e^{(\ln x)} = x.$$

Singular Sturm-Liouville Problem.

(i) $r(a)=0 \Rightarrow$ no boundary condition at $x=a$.

(ii) $r(b)=0 \Rightarrow$ —— " —— at $x=b$

(iii) $r(a)=r(b)=0 \Rightarrow$ No conditions

(iv) $r(a)=r(b) \Rightarrow$ periodic conditions; $y(a)=y(b)$, $y'(a)=y'(b)$.

Then the conclusions of Theorem are true.

Quiz 4.

1. Complet, Sin, Cosine Fourier series
   $[-p, p]$ $[0, p]$ $[0, p]$
   - computation of coefficients
   - extension of sums
   - connections with even and odd functions

2. Complex Fourier Series

   Special attention: connections with real F. series.

3. Eigen functions for boundary problems for $y'' + \lambda y = 0$.