Lecture 16.

General view on Fourier Series. Sect. 12.5

Eigen functions and eigen values on the line.

(1) $\boxed{y'' + \lambda y = 0}$, $y(x)$, $-\infty < x < \infty$

$y_\lambda(x)$

y_λ is an eigen function iff
(i) $y_\lambda(x) \not\equiv 0$
(ii) $y_\lambda'' + \lambda y = 0$.

1) $\lambda \geq 0$, $\lambda = a^2$

$y_a = c_1 \cos ax + c_2 \sin ax$.

2) $\lambda < 0$, $\lambda = -a^2$

$y_a(x) = c_1 e^{ax} + \cancel{c_2} \; c_2 e^{-ax}$

$= d_1 \cosh(ax) + d_2 \sinh(ax)$

$\cosh(ax)' = a \sinh(ax)$
$\sinh(ax)' = a\cosh(ax)$

3) $\lambda = 0$, $y'' = 0$

$y_0(x) = c_1 x + c_0$.

Eigen functions for boundary problems on segments.

Examples. 1. $y'' + \lambda y = 0$, $[0, \pi]$

$y(0) = y(\pi) = 0$ (2 conditions since the equation of 2nd order).

The selection of eigen functions, satisfying to the boundary conditions.

a) $\lambda > 0$, $\lambda = a^2$

$y_a(x) = c_1 \cos(ax) + c_2 \sin(ax)$

$y_a(0) = 0 \Rightarrow c_1 = 0 \Rightarrow y_a(x) = c \sin(ax)$

$y_a(\pi) = 0 \Rightarrow c \neq 0$ ($y_a \equiv 0$ is not an eigenfunction)

$\sin(a\pi) = 0 \Rightarrow a\pi = n\pi \Rightarrow$

$\Rightarrow a = n$, $n = 1, 2, \ldots \Rightarrow$

$y_n(x) = \sin(nx)$, $n = 1, 2, \ldots$, are the eigen functions with the eigen values $\lambda_n = n^2$.

b) $\lambda < 0$, $\lambda = -a^2$

$y_a(x) = c_1 e^{at} + c_2 e^{-at}$

$y_a(0)=0 \Rightarrow c_1+c_2=0 \Rightarrow y_a(x)=c\left(e^{at}-e^{-at}\right),\ c\neq 0$

$y_a(\pi)=0 \Rightarrow e^{a\pi}-e^{-a\pi}=0 \Rightarrow e^{a\pi}=e^{-a\pi} \Rightarrow e^{2a\pi}=1 \Rightarrow$

$\Rightarrow a=0 \Rightarrow c=0$ Contradiction

No eigen functions with $\lambda<0$.

c) $\lambda=0 \Rightarrow y_0(x)=c_1x+c_0$

$y_0(0)=0 \Rightarrow c_0=0,\ y_0(x)=c_1x$

$y_0(\pi)=0 \Rightarrow c_1\pi=0 \Rightarrow c_1=0,\ y_0\equiv 0$ Contradict.

No eigen functions with $\lambda=0$.

Only $y_n(x)=\sin nx$ are e. functions.

$n=1,2,3,\dots$

$\{\sin nx\}$ are base functions for sine F. series on $[0,\pi]$.

Observation: $\{y_n(x)\}$ is the base of the Sine Fourier series

2. $y'' + \lambda y = 0$, $[0, p]$ (2)
$y'(0) = y'(p) = 0$.

Selection of e. functions

a) $\lambda > 0$, $\lambda = a^2$, $y_a(x) = C_1 \cos ax + C_2 \sin ax$

$y_a'(x) = -C_1 a \sin ax + C_2 a \cos ax$

$y_a'(0) = 0 \Rightarrow C_2 = 0$, $y_a(x) = C \cos ax$

$y_a'(p) = 0 \Rightarrow -C a \sin ap = 0$, $C \neq 0, a \neq 0$

$\Rightarrow \sin ap = 0 \Rightarrow ap = n\pi \Rightarrow a = \frac{n\pi}{p}$.

$y_n(x) = \cos\left(\frac{n\pi}{p}x\right)$, $n = 1, 2, \dots$ are eigen functions with the eigen values $\lambda_n = \frac{n^2\pi^2}{p^2}$.

b) $\lambda < 0$, $\lambda = -a^2$

let us use hyperbolic functions (rather than exponential ones).

$y_a(x) = c_1 \cosh ax + c_2 \sinh ax$

$y_a'(x) = a c_1 \sinh ax + a c_2 \cosh ax$

$y_a'(0)=0 \Rightarrow c_2=0 \Rightarrow y_a(x)=c\cosh ax$, $y_a'(x)=ac\sinh a$

$a>0, c\neq 0$

$y_a'(p)=0 \Rightarrow \sinh(ap)=0 \Rightarrow a=0$ contradiction

$\sinh x=0 \Leftrightarrow e^x=e^{-x} \Leftrightarrow e^{2x}=1 \Leftrightarrow x=0.$

No eigen functions with e.value $\lambda<0$,

3) $\lambda=0 \qquad y_0(x)=c_1x+c_0$

$y_0'(0)=0 \Rightarrow c_1=0,\ y_0(x)=c$

So $y_0(x)=1$ is an eigen function with $\lambda=0$.

Unify cases 1) and 3):

$y_n(x)=\cos\left(\frac{n\pi}{p}x\right)$, $n=0,1,2,\ldots$ are eigen functions for the problem (2) with the eigen values $\lambda_n=\frac{n^2\pi^2}{p^2}$. It correspond to the Cosine Fourier series.

Conclusion. Different boundary problems for the equation $y''+\lambda y=0$ correspond to different Fourier series:

Systems of eigen functions are always orthogonal and complete.

A new type of Fourier series.

3) $y'' + \lambda y = 0, \quad [0,\pi]$
$y(0) = y'(\pi) = 0$ (3)

1) $\lambda > 0, \ \lambda = a^2$

$y_a(x) = c_1 \cos ax + c_2 \sin ax$

$y_a(0) = 0 \Rightarrow c_1 = 0, \ y_a(x) = c \sin ax$ (see above).
$c \neq 0, a \neq 0$
(since $y \equiv 0$ isn't an eigen function)

$y_a'(x) = c\,a \cos ax$

$y_a'(\pi) = 0 \Leftrightarrow \cos a\pi = 0 \Leftrightarrow a\pi = \frac{\pi}{2}(2k+1), \ k = 0, 1, 2, \ldots$

$a = \frac{2k+1}{2}$

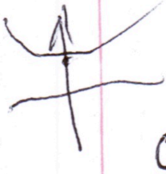
$y_k(x) = \sin\left(\frac{2k+1}{2}x\right), \ k = 0, 1, 2, \ldots$ are eigen functions with the eigen values $\lambda_k = \frac{(2k+1)^2}{4}$.

2) $\lambda < 0, \ \lambda = -a^2$

$y_a(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$

$y_a(0) = 0 \Rightarrow c_1 = 0, \ y_a(x) = c \sinh(ax), \ c \neq 0, \ a \neq 0$

$y_a'(x) = ca \cosh(ax), \ y_a'(\pi) = 0 \Rightarrow \cosh(a\pi) = 0$ (impossible



$\cosh x = \frac{1}{2}(e^{x} + e^{-x}) > 0$, No eigen values $\lambda < 0$.

3) $\lambda = 0$

$y_0(x) = C_1 x + C_0$

$y_0(0) = 0 \Rightarrow C_0 = 0$, $y_0(x) = cx$, $y_0'(x) = c$.

$y_0'(\pi) = 0 \Rightarrow c = 0$, $y_0 \equiv 0$ Contradiction,

$\lambda = 0$ isn't an eigen value.

Only $\left\{ \sin\left(\frac{2k+1}{2} x\right) \right\}$ is the complet system of eigen functions.

Compare with Sine Fourier series

$\{\sin mx\}$

Which boundary problem corresponds to the complet Fourier Series?

$y'' + \lambda y = 0$, $[-\pi, \pi]$

$y(-\pi) = y(\pi)$
$y'(-\pi) = y'(\pi)$ (periodical condition).

1) $\cos(nx)$, $\sin(nx)$, $\lambda_n = n^2$
satisfy to this condition
2 indep. eigen functi

2) $\lambda < 0$
No periodic Eigen functions!

3) $\lambda = 0$ $y_0(x) \equiv 1$ is periodic