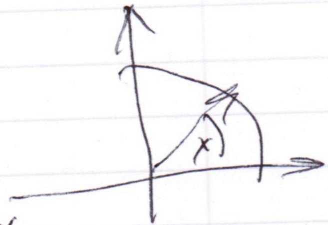
Lecture 15. (Section 12.4).

Complex Fourier Series.

Euler's formula:



$$e^{ix} = \cos x + i \sin x.$$

$$e^{i(x_1+x_2)} = e^{ix_1} e^{ix_2}.$$

[compatible with series]

$e^{2\pi i} = 1$, $e^{\pi i} = -1$, $e^{n\pi i} = (-1)^n$

$e^{\frac{\pi}{2} i} = i$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$[-p, p]$. $\exp\left(i \frac{n\pi x}{p}\right)$ is a complet orthogonal system

$-\infty < n < \infty$

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \exp\left(i \frac{n\pi x}{p}\right)$$

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) \exp\left(-i \frac{n\pi x}{p}\right) dx.$$

Example.

$f(x) = \exp(2x)$, $[-\pi, \pi]$. $\exp(inx)$ $-\infty < n < \infty$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(x(2-in))\,dx =$$

$$= \frac{1}{2\pi} \frac{1}{2-in} \exp(x(2-in)) \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(2-in)} \left(\exp(2\pi - i\pi n) - \exp(2\pi + i\pi n) \right)$$

$e^{i\pi n} = e^{-i\pi n} = (-1)^n$

$$= \frac{(-1)^n}{2\pi(2-in)} (\exp 2\pi - \exp(-2\pi))$$

$$= \frac{(-1)^n}{\pi(2-in)} \sinh 2\pi$$

$$\exp(2x) = \frac{\sinh 2\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2-in} e^{inx}$$

$-\pi < x <$

$c_0 = \frac{\sinh(2\pi)}{2\pi}$

$$\frac{1}{2-in} = \frac{2+in}{4+n^2}$$

Properties.

f - even $\sim$ $c_n = c_{-n}$

f - odd $\sim$ $c_n = -c_{-n}$ (including $c_0 = 0$)

$$\exp\left(i\,\frac{n\pi x}{p}\right) = \cos\left(\frac{n\pi x}{p}\right) + i\sin\left(\frac{n\pi x}{p}\right)$$

$$\exp\left(-i\,\frac{n\pi x}{p}\right) = \cos\left(\frac{n\pi x}{p}\right) - i\sin\left(\frac{n\pi x}{p}\right)$$

$a_n = c_n + c_{-n}$, $a_0 = 2c_0$

$b_n = i(c_n - c_{-n})$

$$c_n = \frac{a_n - ib_n}{2} \qquad c_0 = \frac{a_0}{2}$$

$$c_{-n} = \frac{a_n + ib_n}{2} \qquad c_{-n} = \overline{c_n}$$

(for real f).

We can transform complex Fourier series at real ones and back!

$f(x) = \exp(2x)$, $[-\pi, \pi]$

$$a_n = \frac{\sinh(2\pi)}{\pi}(-1)^n\left(\frac{1}{2-in} + \frac{1}{2-in}\right) =$$

$$\underbrace{(-1)^n = (-1)^{-n}} = \frac{4\sinh 2\pi}{\pi}\frac{(-1)^n}{n^2+4}$$

$$b_n = \frac{\sinh(2\pi)}{\pi}(-1)^n i\left(\frac{1}{2-in} - \frac{1}{2in}\right) =$$

$$= \frac{\sinh(2\pi)}{\pi}(-1)^n\frac{2n}{4+n^2}$$

$$\exp(2x) = \frac{\sinh(2\pi)}{\pi}\left(\frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+4}\cos nx\right.$$

$-\pi < x < \pi$

$$+2\sum_{m=0}^{\infty}\frac{(-1)^m m}{m^2}\sin mx$$