MATH 421.03 ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 3

1. (100 points) a) Give the definition of orthogonal functions.

b) Prove that the given system of functions

$$1, \cos 2x, \cos 4x, \cos 6x, \ldots, \cos 2nx, \ldots$$

is orthogonal on $[0, \pi/2]$.

c) Find the norms of each of these functions.

d) Are functions $\cos(2\pi kx)$ orthogonal on $[0, \pi/2]$? Give a detailed explanation.

a) $f \perp g \Leftrightarrow (f,g) = \int_a^b f(x)\, g(x)\, dx = 0.$

b) $\int_0^{\pi/2} \cos(2kx) \cos(2\ell x)\, dx = \frac{1}{2}\int_0^{\pi} \cos(ky) \cos(\ell y)\, dy$

$k \neq \ell$

$\cos a \cos b = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right).$ $\quad = \frac{1}{4}\left(\int_0^{\pi} \cos((k+\ell)y) + \cos((k-\ell)y)\right) dy$

$= \frac{1}{4}\left(\frac{\sin(k+\ell)y}{k+\ell} + \frac{\sin(k-\ell)y}{k-\ell}\right)\Big|_0^{\pi} = 0$, since $\sin n\pi = 0$

c) $\|f\|^2 = \frac{1}{2}\int_0^{\pi} \cos^2 ky\, dy = \frac{1}{4}\int_0^{\pi} (\cos(2ky) + 1)\, dy = \begin{cases} \pi/4, & k \neq 0 \\ \pi/2, & k = 0. \end{cases}$

d) $(1, \cos 2\pi x) = \int_0^{\pi/2} \cos(\cancel{\cos}\ 2\pi x)\, dx = \frac{1}{2\pi} \sin(2\pi x)\Big|_0^{\pi/2} \neq 0.$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

2. (100 points) a) Expand the function on $(-\pi, \pi)$:

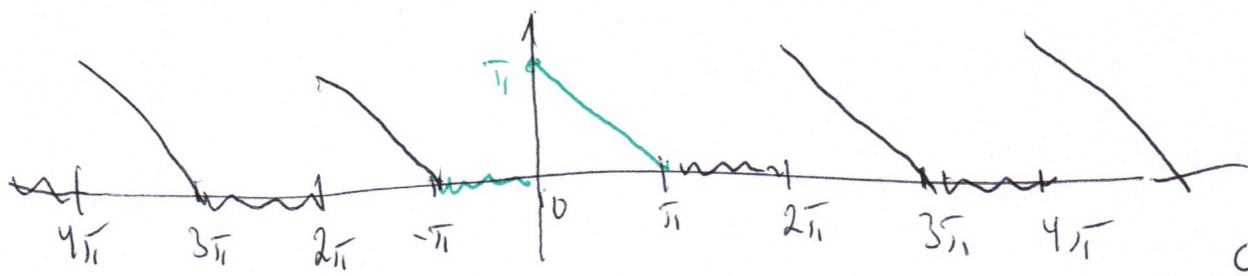
$$f(x) = 0, -\pi < x < 0, f(x) = \pi - x, 0 < x < \pi$$

in the Fourier series.

b)Specify the numerical series which are the results of evaluations for $x = 0, \pi/2$.

c)Describe the sum of these series for all $-\infty < x < \infty$. Give explanations and sketch the graphs.

d) Consider the Fourier series which keeps only the terms with the sinuses of our series. What is the sum of this series? Sketch the graph of this function.



c)

$$a_0 = \frac{1}{\pi}\int_0^\pi (\pi - x)\,dx = \boxed{\frac{\pi}{2}},$$

$$a_n = \frac{1}{\pi}\int_0^\pi (\pi - x)\cos nx\,dx = \frac{1}{\pi}\left(\pi\int_0^\pi \cos nx\,dx - \int_0^\pi x\cos nx\,dx\right)$$

$n \neq 0$

$$= \frac{1}{\pi}\left(\underbrace{\frac{\pi}{n}\sin nx\Big|_0^\pi}_{0} - \underbrace{\frac{x}{n}\sin nx\Big|_0^\pi}_{0} + \frac{1}{n}\int_0^\pi \sin nx\,dx\right)$$

$$= -\frac{1}{\pi n^2}\cos nx\Big|_0^\pi = \boxed{\frac{1}{\pi n^2}\left(1 - (-1)^n\right)},$$

$$b_n = \frac{1}{\pi}\left(\pi\int_0^\pi \sin nx\,dx - \int_0^\pi x\sin nx\,dx\right) = \frac{1}{\pi}\left(-\frac{\pi}{n}\cos nx\Big|_0^\pi + \right.$$

$$\left. + \frac{x}{n}\cos nx\Big|_0^\pi - \underbrace{\frac{1}{n}\int_0^\pi \cos nx\,dx}_{0}\right) = \frac{1}{\pi}\left(\frac{\pi(1-(-1)^n}{n} + \frac{\pi}{n}(-1)^n\right)$$

$$= \boxed{\frac{1}{n}}$$

$$f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2}\cos nx$$
$-\pi < x < \pi$

$$+ \sum_{m=1}^{\infty} \frac{1}{m} \sin mx .$$

b) $x=0$

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} \quad\Big|\quad \frac{\pi^2}{8} = \sum_{\text{odd } n} \frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \dots$$

"jump"

$x = \frac{\pi}{2}$

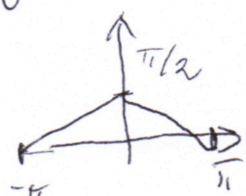
$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \quad\Big|\quad \frac{\pi}{4} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - - - -$$

n even $-$ $a_n = 0$

n odd, $\cos n\frac{\pi}{2} = 0.$

m ~~odd~~ even $\sin m\frac{\pi}{2} = 0$

d) $f = f_{even} + f_{odd}$



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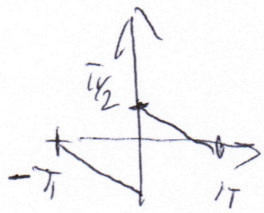

$$f_e = \frac{\pi - |x|}{2}$$

Sum of sinuses $-$ f_{odd}

$$f_{odd} = \cancel{\frac{(\pi-|x|)}{2}\operatorname{sgn} x}$$
$$= \begin{cases} \frac{\pi - x}{2}, & x>0 \\ \frac{-\pi - x}{2}, & x<0 \end{cases}$$
$$= \left(\frac{\pi - |x|}{2}\right)\operatorname{sgn} x$$

MATH 421.08 ADVANCED CALCULUS FOR ENGINEERING. SPRING 2014. QUIZ 3

1. (100 points) a) Give the definition of orthogonal functions.

b) Prove that the given system of functions

$$\sin 2\pi x, \sin 4\pi x, \sin 6\pi x, \ldots, \sin 2n\pi x, \ldots$$

is orthogonal on $[0, 1/2]$.

c) Find the norms of each of these functions.

d) Are functions $\sin(k\pi x)$ orthogonal on $[0, 1/2]$? Give a detailed explanation.

a) $$\int_0^{1/2} \sin(2\pi n x)\sin(2\pi m x)\,dx = \frac{1}{2}\int_0^{1/2}\left[\cos(2\pi(n-m)x - \cos 2\pi(n+m)x\right].$$
$m \neq n$

$$= \frac{1}{2}\left[\frac{\sin 2\pi(n-m)x}{2\pi(n-m)} - \frac{\sin 2\pi(n+m)x}{2\pi(n+m)}\right]\Bigg|_0^{1/2}$$

$$= \frac{1}{2}\left[\frac{\sin \pi(n-m)}{2\pi(n-m)} - \frac{\sin \pi(n+m)}{2(n+m)}\right] = 0$$

b) $n = m$

$$\|f\|^2 = \int_0^{1/2} \sin^2(2\pi n x)\,dx = \frac{1}{2}\int_0^{1/2}(1-\cos(4\pi n x))\,dx$$

$$= \frac{1}{2}\int_0^{1/2} dx = \frac{1}{4} \qquad \|\cos(2\pi n x)\| = \boxed{\frac{1}{2}}$$

c) $$(\sin \pi x, \sin 2\pi x) = \frac{1}{2}\int_0^{1/2}(\cos \pi x - \cos 3\pi x) =$$

$$= \frac{1}{2}\left[\frac{\sin \pi x}{\pi} - \frac{\sin 3\pi x}{3\pi}\right]\Bigg|_0^{1/2} = \frac{1}{2}\left[\frac{\sin\frac{\pi}{2}}{\pi} - \frac{\sin\frac{3\pi}{2}}{3\pi}\right]$$

$$= \frac{1}{2\pi} + \frac{1}{3\pi} \neq 0$$

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c) $(\sin k\pi x, \sin l\pi x) =$

$$= \frac{1}{2}\left[\frac{\sin (k-l)\pi/2}{\pi(k-l)} - \frac{\sin(k+l)\pi/2}{\pi(k+l)}\right]$$

k, l have the same eveness ($k-l$ - even)

$(\, , \,) = 0$

No if not.

2. (100 points) a) Expand the function on $(-1, 1)$:

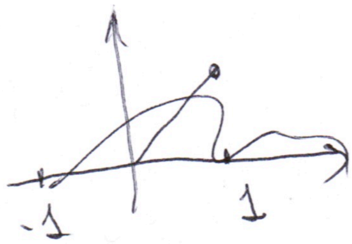
$$f(x) = x, -1 < x < 0, f(x) = 1 - x, 0 < x < 1$$

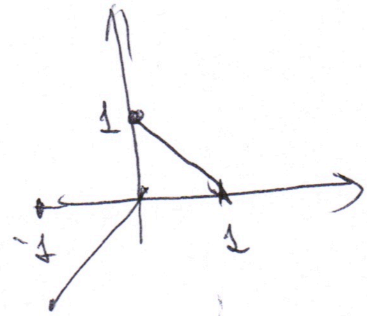
in the Fourier series.

b)Specify the numerical series which are the results of evaluations for $x = 0, 1/2$.

c)Describe the sum of these series for all $-\infty < x < \infty$. Give explanations and sketch the graphs.

d) Consider the Fourier series which keeps only the terms with the cosinuses of our series. What is the sum of this series? Sketch the graph of this function.



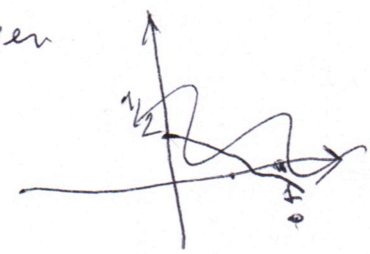


ext

$\cos(n\pi x)$, $\sin(m\pi x)$

f_{even}

$p = 1$



$$a_0 = \int_{-1}^{0} x\,dx + \int_0^1 (1-x)\,dx = -\frac{1}{2} + \frac{1}{2} = 0$$

$$a_n = \int_{-1}^{0} x \cos(n\pi x)\,dx + \int_0^1 (1-x)(\cos n\pi x)\,dx$$

$$= -2 \int_0^1 y \cos(n\pi y)\,dy + \int_0^1 \cos(n\pi x)\,dx$$

$$= \underbrace{\frac{1}{n\pi} \sin n\pi x \Big|_0^1}_{0} - \underbrace{2\frac{1}{n\pi} y \sin n\pi y \Big|_0^1}_{0} + \frac{2}{n\pi} \int_0^1 \sin n\pi y\,dy$$

$$= \frac{2}{n^2\pi^2} \cos n\pi y \Big|_0^1 = \boxed{\frac{2}{n^2\pi^2}\left(1 - (-1)^n\right)}$$

$a_n = 0$
n even

$$b_m = \int_{-1}^{0} x \sin m\pi x + \int_0^1 (1-x) \sin m\pi x \, dx$$

$$= \int_0^1 \sin(\pi m x \, dx) = -\frac{1}{\pi m} \cos(\pi m x)\Big|_0^1 = \frac{1-(-1)^m}{\pi m}$$

$$f(x) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos(n\pi x) + \cancel{1-(-1)^n}$$

$$+ \frac{1}{\pi} \sum_{m=1}^{\infty} \cancel{\sin(n\pi} \frac{1-(-1)^m}{m} \sin(n\pi \cancel{x}).$$

b) $x = 0$

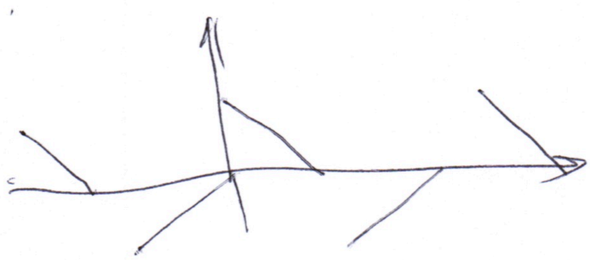
$$\frac{1}{2} = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \quad \Big| \quad \frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \dots$$

$\cancel{x = \frac{\pi}{2}}$ $x = \frac{1}{2}$ $\cos\left(\frac{(2k+1)\pi}{2}\right) = 0$

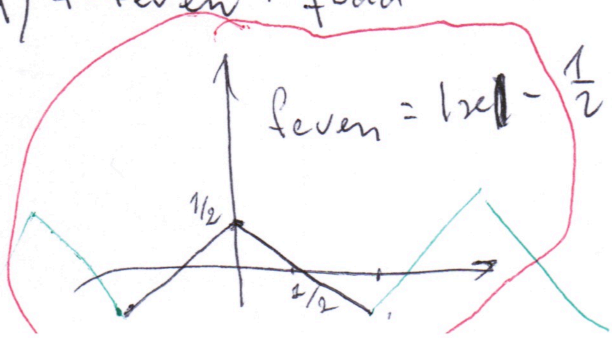
$$\sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$$

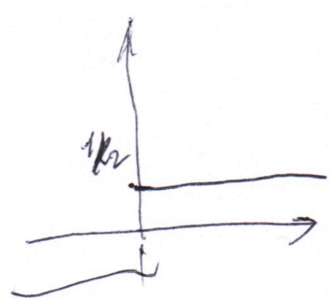
$$\frac{1}{2} = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \quad \Big| \quad \frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

c)



d) $f = f_{even} + f_{odd}$





$f_{odd} = \frac{1}{2} \operatorname{sgn} x.$