[illegible]
13 - 1

Lecture 13

Fourier series on $[-p, p]$ $\quad$ $2p$-periodic

Orthog. system $\cos \frac{\pi n x}{p}, \sin \frac{\pi n x}{p}$

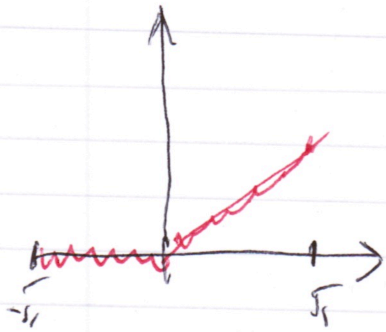
$$f(x) = \frac{a_0}{2} + \sum \left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right)$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p} \, dx$$

$$b_m = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p} \, dx$$

Lecture 13.

Ex. $f(x) = \cancel{x} \frac{1}{2}(x+|x|) = \begin{cases} x, & 0<x<\pi \\ 0, & -\pi<x<0 \end{cases}$



Take the sum of Fourier series.

Each f on $[-p,p]$ can be present as

$$f(x) = f_{even}(x) + f_{odd}(x)$$

$$f_{even}(x) = \frac{f(x)+f(-x)}{2}$$

$$f_{odd}(x) = \frac{f(x)-f(-x)}{2}$$

Verify for our example

$$f(x) = \underbrace{\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}}_{=\frac{|x|}{2}}$$

$$+ \underbrace{2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin nx}{n}}_{=\frac{x}{2}}.$$

Cosine and Sine Fourier Series.

Sect. 12

$\left\{\cos \frac{n\pi x}{p}, \sin \cancel{mx}\, \frac{m\pi x}{p}\right\}$ is complet on $[-p,p]$

Each of

$\left\{\cos \frac{n\pi x}{p}\right\}$ and $\left\{\sin \frac{m\pi x}{p}\right\}$

are complet on $[0,p]$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p} \quad \text{[cosine seri}$$

$$= \cancel{a_n} \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{2} \quad \text{[sine series]}$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi x}{p} \, dx$$

$$b_m = \frac{2}{p} \int_0^p f(x) \sin \frac{m\pi x}{p} \, dx$$

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Fourier S. on $[-p,p]$ for even functions $\Longleftrightarrow$ Cosine F. Series on $[0,p]$.

-11- for odd functions $\Longleftrightarrow$ Sine F series on $[0,p$

The same function $f(x)$ on $[0,p]$ can be expand at both Cosine and Sine Fourier series.

Examples.

$p=\pi \qquad f(x)=x$

Even extension $|x|$

Odd one x

Cosine F. series

$$x=\frac{\pi}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cos nx$$

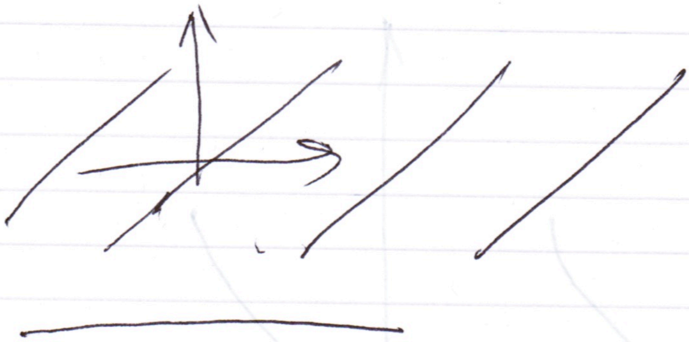
$0<x<\pi$

Sine F. series

$$x=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$$

13.5

Extensions of series on $(-\infty, \infty)$ are different:

$$f_1(x) = \cancel{f(x)}\, x^2, \quad -\pi < x < \pi$$

$$\underset{n\neq 0}{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx = \underbrace{\frac{1}{\pi}\left[\frac{x^2 \sin nx}{n}\right]_{-\pi}^{\pi}}_{=0}$$

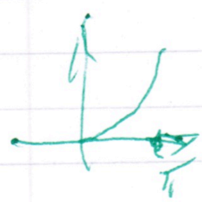
$$- \frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin nx$$

$$= \frac{2(-1)^{n+1}}{n}\pi \quad \text{(old computation)}$$

$$a_n = \frac{4(-1)^n}{n^2}, \quad n \neq 0$$

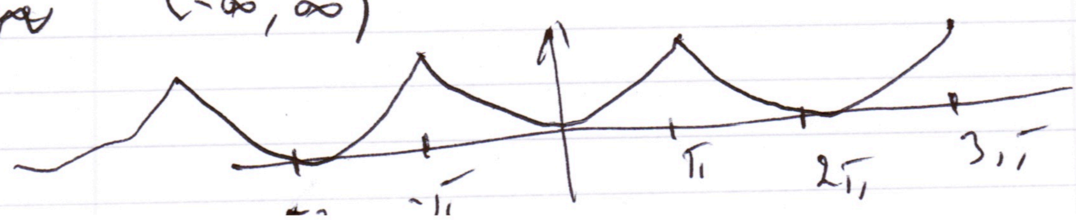
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^2}{3}$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$



Corollary. It's the Cosine Fourier series on $[0, \pi]$

On $(-\infty, \infty)$



$f_2(x) = x^2, \quad [0, \pi]$

Find Sine F. series

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx = -\frac{2x^2 \cos nx}{\pi n}\Big|_0^{\pi}$$

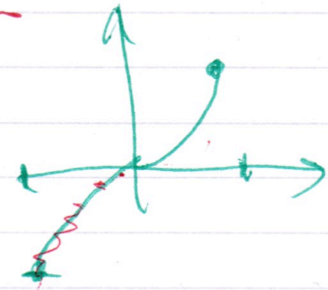
$$+ \frac{4}{\pi n}\int_0^{\infty} x \cos nx \, dx$$ ← already computed

$$= \frac{2\pi(-1)^{n+1}}{n} + \frac{4((-1)^n - 1)}{\pi n^3}$$

$$x^2 = \sum_{n=1}^{\infty}\left(\frac{2\pi(-1)^{n+1}}{n} + \frac{4((-1)^n-1)}{\pi n^3}\right)\sin nx,$$

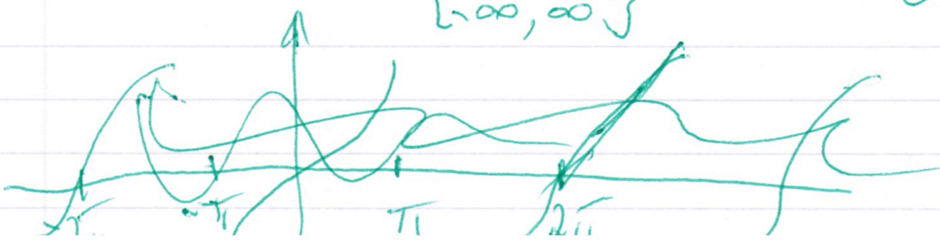
$0 < x < \pi$

Odd extension on $[-\pi, \pi]$



$$\operatorname{sgn} x \cdot x^2 = \begin{cases} x^2 & x > 0 \\ -x^2 & x < 0 \end{cases}$$

$[-\infty, \infty]$



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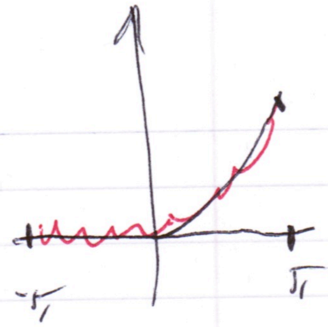
$$f(x) = \begin{cases} x^2, & 0 < x < \pi \\ 0, & -\pi < x < 0 \end{cases}$$



$$a_n = \frac{2(-1)^n}{n^2}, \ n=0, \quad a_0 = \frac{\pi^2}{3}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx = -\left.\frac{x^2 \cos nx}{\pi n}\right|_0^{\pi}$$

$$+ \frac{2}{\pi n}\int_0^{\pi} x \cos nx \, dx =$$

$$= \frac{\pi(-1)^{n+1}}{n} + \frac{2((-1)^n - 1)}{\pi n^3}.$$

$$\boxed{\begin{aligned} f(x) &= \frac{1}{2}\left(f_1(x) + f_2(x)\right) \\ &= f_{even} + f_{odd} \end{aligned}}$$

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

$$+ \sum_{m=1}^{\infty}\left(\frac{\pi(-1)^{m+1}}{m} + \frac{2((-1)^m - 1)}{\pi m^3}\right)\sin mx$$

Examples of Cosine and Sine Series:

$[0,\ell]$

$$x = \frac{2\ell}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin nx}{n}\right.$$

$$x = \frac{\ell}{2} - \frac{4\ell}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos\left((2k+1)\frac{\pi x}{\ell}\right)}{(2k+1)^2}$$

$$= \frac{\ell}{2} - \frac{2\ell}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}\cos\left(\frac{n\pi x}{\ell}\right)$$

$$1 = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(\frac{(2k+1)\pi x}{\ell}\right).$$

$1=1$

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Sine F. series for $f(x) = \cos x$ on $[0,\pi]$

Cosine F. series $\cos x = \cos x$

$$b_n = \frac{2}{\pi}\int_0^\pi \cos x \sin nx \, dx = \frac{1}{\pi}\int_0^\pi \sin((n+1)x) + \sin((n-1)x)\,dx$$

$$= \frac{1}{\pi}\left(-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right)\Bigg|_0^\pi$$

$$-\frac{1}{n+1}\left((-1)^{n+1}-1\right)$$

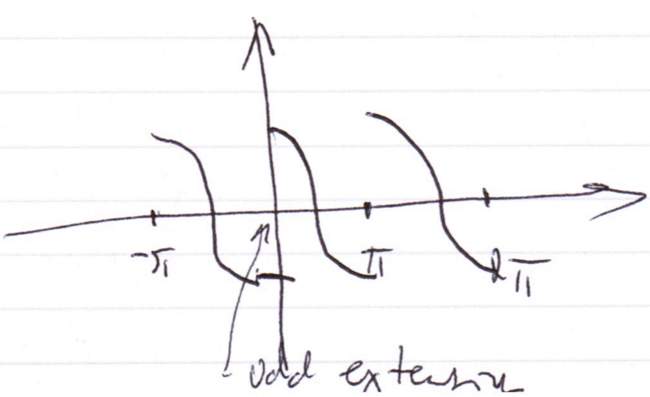
$$\underset{n\geq 2}{=} \begin{cases} \frac{2}{\pi}\left(\frac{1}{n+1}+\frac{1}{n-1}\right) & n \text{ even} \\ 0 & \text{odd} \end{cases}$$

$$= \begin{cases} \frac{4}{\pi}\frac{n}{n^2-1}, & n \text{ even} \\ 0 & n>1 \text{ odd} \end{cases}$$

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin 2x \, dx = 0$$

$$\cos x = \frac{4}{\pi}\sum_{n \text{ even}} \frac{n}{n^2-1} \sin nx.$$

$0 < x < \pi$



π-periodic

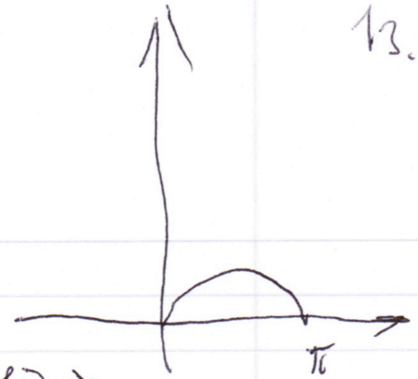
9.1

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$

$$\cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$

$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

13.



$f(x) = \sin x, \quad 0 < x < \pi$

Sine series is trivial. $\sin x = \sin x$

Fine Cosine F. Series

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx = \frac{4}{\pi}$$

$$\underset{n \neq 0}{a_n} = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx =$$

$$= \frac{1}{\pi} \int_0^{\pi} (\sin (n+1)x - \sin (n-1)x) \, dx$$

$$n > 1 \quad = \frac{1}{\pi} \left(- \frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1} \right) \Bigg|_0^{\pi}$$

$$= \begin{cases} \frac{2}{\pi} \left(\frac{1}{n+1} - \frac{1}{n-1} \right) = - \frac{4}{\pi} \frac{1}{n^2 - 1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$a_1 = 0$

Other examples:

$$\sin x = \frac{2}{\pi}\left(1 - 2\sum_{\substack{n \text{ even} \\ n>0}} \frac{\cos nx}{n^2-1}\right) =$$

$0<x<\pi$

$$= \frac{2}{\pi}\left(1 + \sum_{n>0 \text{ even}} \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\cos nx\right).$$

$$x=0 \qquad \frac{1}{2} = \sum_{n>0 \text{ even}} \frac{1}{n^2-1}$$

$$0 = \left(1 + \frac{1}{3} - 1 + \frac{1}{5} - \frac{1}{3} \dots\right.$$