Lecture 12 (~~Select~~ Section 12.2).

$[-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$a_n = \frac{(f, \cos nx)}{\|\cos nx\|^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 0, 1, \ldots$$

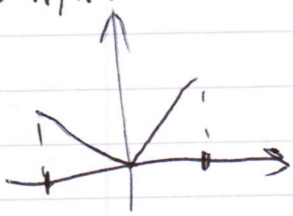
$$b_n = \frac{(f, \sin nx)}{\|\sin nx\|^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \ldots$$

Convergence for continuous functions (pointwise).

For piecewise continuous functions

$$\to \frac{f(x-0) + f(x+0)}{2}$$

2) $f(x) = |x|$ $[-\pi, \pi]$



$b_n \equiv 0$ (Why?) even

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$\boxed{n > 0}$

$$= \frac{2}{\pi} \left(\underbrace{\frac{x \sin nx}{n} \Big|_0^{\pi}}_{=0} - \frac{1}{n} \int_0^{\pi} \sin nx \, dx \right)$$

$$= \frac{2}{\pi n^2} \cos nx \Big|_0^{\pi} = \frac{2((-1)^n - 1)}{\pi n^2} = \frac{4}{\pi n^2} \begin{cases} 0 & n\text{-even} \\ -1 & n\text{-odd} \end{cases}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi$$

$$\underset{-\pi < x < \pi}{|x|} = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}$$

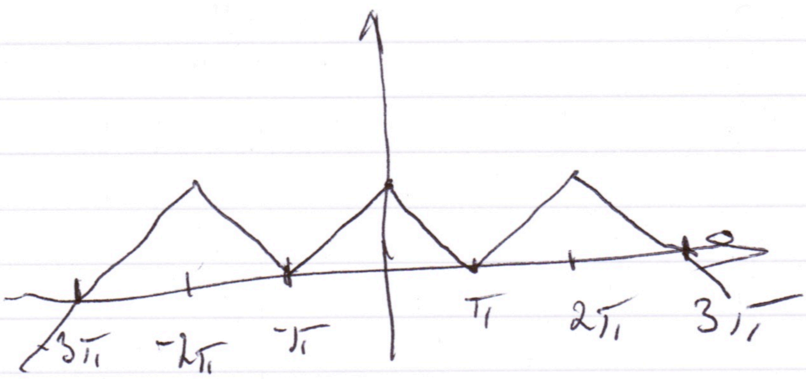
odd $n = 2k + 1$.

Special points

$x=0$ $$\frac{\pi}{2} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = 1+\frac{1}{9}+\cancel{\frac{1}{16}}+\frac{1}{25}+\dots$$

2π- periodic extension



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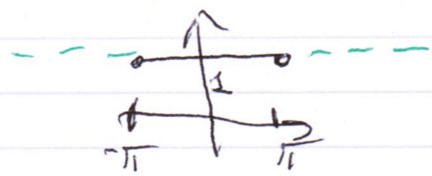
3. Fourier series for $f(x)=1$

$1=1$

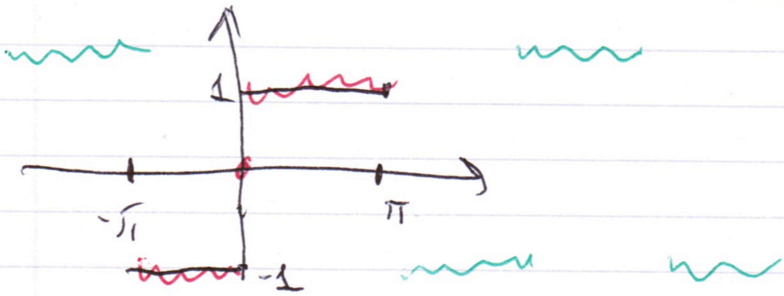


If f is an element of the base its Fourier series has only one term

$\sin 3x = \sin 3x$

12.4

4) $f(x) = \operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases} \quad [-\pi, \pi]$



f is odd $\Rightarrow a_n = 0$ $\quad f(x) = -f(-x)$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{2}{\pi}\left(-\frac{\cos nx}{n}\Big|_0^{\pi}\right)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn} x \sin nx\, dx \qquad = \frac{2}{\pi n}\left(1-(-1)^n\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd} \end{cases}$$

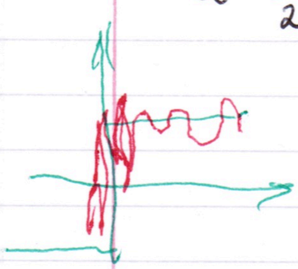
$$\operatorname{sgn} x = \frac{2}{\pi}\sum_{n=1}^{\infty} \cancel{\sin}\, \frac{(1-(-1)^n)}{n} \sin nx =$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$$

only odd terms.

$x = 0 \qquad 0 = 0$

$x = \frac{\pi}{2} \qquad 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right)$



$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \dots$$

We already have this series.