Lecture 10. Sect. 12.2

① Why orthogonal systems are convenient?

Good formulas for coefficients of series.

$\{\varphi_1, \ldots, \varphi_n, \ldots\}$ is an orthogonal system. and

$$f = \sum_{n=1}^{\infty} a_n \varphi_n \qquad (*)$$

Then
$$a_n = \frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)}. \qquad (**)$$

Just take the inner product with φ_n (*).
On the right only one term survives!

$$(\varphi_n, f) = a_n (\varphi_n, \varphi_n).$$

We have on $[-\pi, \pi]$ the orthogonal system

$$\{ \sin nx,\ n=1,\dots;\ \cos mx,\ m=0,1,\dots \}.$$

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (**)$$

Fourier series (or trigonometric series)

Then $a_n = \dfrac{(f, \cos nx)}{\pi}$, $n = 0, 1, 2, \dots$

$b_m = \dfrac{(f, \sin mx)}{\pi}$, $m = 1, \dots$

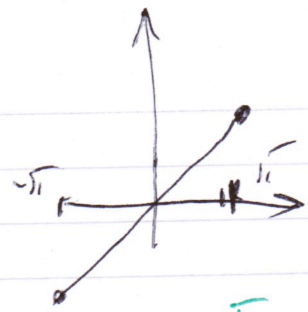
Why we take $\frac{a_0}{2}$?

If we have a continues function $f(x)$, $x \in [-\pi, \pi]$ the we can compute a_n, b_m consider the series $(**)$

It will pointwise convergent to $f(x)$ (without proof).
[at each point]

$[-\pi,\pi]$

Examples.



1. $f(x) = x$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \quad \Big| \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \overset{\text{odd!}}{\cos} nx \, dx = 0 \quad \forall n$$

$n = 1, 2, \ldots$

$$= \frac{1}{\pi}\left(- \frac{x \cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\underbrace{\int_{-\pi}^{\pi} \cos nx \, dx}_{(1, \cos nx) = 0} \right.$$

$$= \frac{1}{\pi}\Big(- \frac{\pi}{n}\big(\underbrace{\cos n\pi}_{(-1)^n} + \cos(-n\pi)\big)\Big) = \frac{2(-1)^{n+1}}{n}$$

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$

$-\pi < x < \pi$

$$= 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right)$$

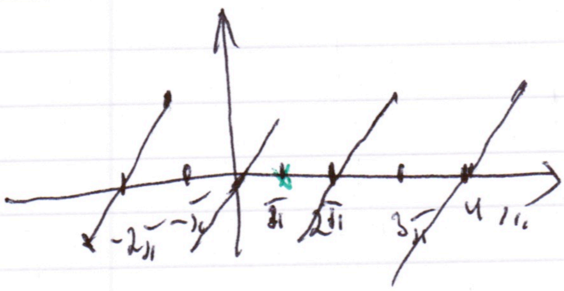
Special points!

$x = 0 \qquad 0 = 0$

$$x = \frac{\pi}{2} \qquad \frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

n even $\quad \sin k\pi = 0.$

Let us consider this series on $(-\infty,\infty)$.
It'll be convergent to $f(x)$ (2π-periodic)



Remark. Fourier series is convergent not only for continuous functions but also for piecewise continuous (as f at this example).

$\rightarrow \dfrac{f(x-0)+f(x+0)}{2}$ at all x.

A examplee the sum $=0$ for $x=(2k+1)\pi$