MATH 3002. INTRODUCTION TO
MATHEMATICAL REASONING.
FALL 2015.
QUIZ 9

1. (25 points) 1) Give the predicative definition that a sequence has a limit if $n \to \infty$; has no limit.

2) Find limits if $n \to \infty$ of the following sequences, if they exist, and prove the existence or non existence using directly the definitions:

$$\frac{1}{n+3}; 3^{-n}; n^2.$$

1) Convergent:

$(\exists L)(\forall \varepsilon)\ \varepsilon > 0 \Rightarrow \{(\exists N)(\forall n)\ n > N \Rightarrow |x_n - L| < \varepsilon\}.$

$(\forall L)(\exists \varepsilon)\ \varepsilon > 0 \wedge \{(\forall N)(\exists n)\ n > N \wedge |x_n - L| \geq \varepsilon\}.$

2) $\lim\limits_{n \to \infty} \frac{1}{n+3} = 0.$

Analysis: $|x_n - L| = \frac{1}{n+3} < \varepsilon \Leftrightarrow n + 3 > \frac{1}{\varepsilon}$. Take $N(\varepsilon) = \left[\frac{1}{\varepsilon}\right] - 3$

Proof: $n > N(\varepsilon) \Rightarrow n+3 > \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{n+3} < \varepsilon$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2) $x_n = 3^{-n}$. $\lim\limits_{n\to\infty} 3^{-n} = 0$.

Analysis: $|x_n - L| = 3^{-n} < \varepsilon \iff -n\ln 3 < \ln \varepsilon \iff$

$$\iff n > -\frac{\ln \varepsilon}{\ln 3}.$$

3) $x^n = n^2$ disverges.

Take any L, N and $\varepsilon = 1$ (independent of L).

$(\exists n)$ Take any n which

(i) $n > N$;

(ii) $n > L + 1$.

$$n^2 - (L+1) > n - (L+1) > 1 \Rightarrow |x_n - L| > \varepsilon = 1$$

Disvergence is proved.

2. (25 points) The same assignment as in Problem 1 but you can apply some theorems on limits:

$$\frac{2n^3+3n^2+5}{5n^3-2n+1};\ \frac{n-1}{n^2+1};\ \frac{(-1)^n n}{n+1}.$$

1) $\lim\limits_{n\to\infty} \dfrac{2n^3+3n^2+5}{5n^3-2n+1} = \lim\limits_{n\to\infty} \dfrac{2+\frac{3}{n}+\frac{5}{n^3}}{5-\frac{2}{n^2}+\frac{1}{n^3}} = \dfrac{2}{5}$,

2) $\lim\limits_{n\to\infty} \dfrac{n-1}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}-\frac{1}{n^2}}{1+\frac{1}{n^2}} = \dfrac{0}{1} = 0.$

3) $x_n = \dfrac{(-1)^n n}{n+1}$ disveiges!

subsequence $y_m = x_{2m+1} \to -1$

$z_p = x_{2p} \to 1.$

3. (25 points) Prove that the limit of the sum of 2 sequences is the sum of its limits.

see Lecture 26

4. (25 points) Prove that if to change a finite numbers of terms in a convergent sequence the limit will no change.

Let $\lim_{n\to\infty} x_n = L$.

Let we change a finite number of x_i ($\{y_n\}$ – new seq)

$(\exists K)(\forall n)\ n > K \Rightarrow y_n = x_n$.

~~If $(\forall n)\ n > N \Rightarrow$~~ We can take for ~~each~~ $(\forall \varepsilon)$ $N(\varepsilon) > K$. So

$(\forall \varepsilon)\ \varepsilon > 0 \Rightarrow \{(\exists N)\ N > K \wedge ((\forall n)\ n > N \Rightarrow |x_n - L| < \varepsilon)$

But for such n $x_n = y_n$ and it means

$\lim_{n\to\infty} y_n = L$.