MATH 3002. INTRODUCTION TO MATHEMATICAL REASONING.

FALL 2015.

QUIZ 8

1. (50 points) 1) Prove by the method of mathematical induction that

$$1) S_n = \frac{1}{1\cdot 2\cdot 3} + \frac{2}{1\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots + \frac{1}{(n-2)\cdot(n-1)\cdot n} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n-1)\cdot n}\right];$$

$$2) (2n)! < 2^{(2n)}(n!)^2.$$

1) (i) $S_3 = \frac{1}{6} = \frac{1}{6}$

(ii) $S_k = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(k-1)\cdot k}\right] \Rightarrow S_{k+1} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{k(k+1)}\right]$

$k \geq 3$

$$P \Rightarrow S_{k+1} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(k-1)k}\right] + \frac{1}{(k-1)k(k+1)} =$$

$$= \frac{1}{4} - \left(\frac{1}{2(k-1)k} - \frac{1}{(k-1)k(k+1)}\right) =$$

$$= \frac{1}{4} - \frac{1}{2}\left(\frac{k+1-2}{(k-1)k(k+1)}\right) =$$

$$= \frac{1}{4} - \frac{1}{2}\left(\frac{1}{k(k+1)}\right)$$

2) (i) $n=1$

$$2! < 2^2$$

$$P \Rightarrow Q$$

(ii) $$(2k!) < 2^{(2k)}(k!)^2 \overset{?}{\Rightarrow} (2(k+1)!) < 2^{(2k+2)}((k+1)!)^2$$

$$P \Rightarrow (2(k+1)!) < 2^{(2k)}(k!)^2 (2k+1)(2k+2) <$$

$$2^{(2k)}(k!)^2 \cdot 4(k+1)^2 = 2^{(2k+2)}(k+1)!$$

2. (25 points) We define a function recursively for all positive integers by $f(1) = 1, f(2) = 5$ and for $n > 2, f(n+1) = f(n) + 2f(n-1)$. Show that $f(n) = 2^n + (-1)^n$.

(i) $n=3$, $f(\cancel{4}3) = 7 = 2^3 - 1$. True

(ii) $f(k) = 2^k + (-1)^k \overset{?}{\Rightarrow} f(k+1) = 2^{k+1} + (-1)^{k+1}$

For all $l \leq k$.

$f(k+1) = f(k) + 2f(k-1) = 2^k + (-1)^k + 2 \cdot 2^{k-1} + 2(-1)^{k-1}$

$= 2^{k+1} + (-1)^{k+1}$.

$(-1)^k = \cancel{} - (-1)^{k+1}$

$(-1)^{k-1} = (-1)^{k+1}$

3. (25 points) Show that any amount of postage that is an integer number of cents greater than 53 cents can be formed using just 7-cent and 10-cent stamps. (Prove by induction).

(i) $54 = 2 \cdot 7 + 4 \cdot 10$

(ii) $k = 7a + 10b \overset{?}{\Rightarrow} (\exists \tilde{a})(\exists \tilde{b})\ k+1 =$
$= 7\tilde{a} + 10\tilde{b}$.
$k \geq 54$

1) Let $b \geq 2$. Then $\tilde{b} = b - 2$, $\tilde{a} = a + 3$
$\cancel{[10 \cdot 2 = 7 \cdot 3 + 1}$

$10 \cdot 2 + 1 = 7 \cdot 3$

2) Let $b \leq 1$ (0 or 1). The $7a \geq 44$
$a \geq 7 \Rightarrow \tilde{a} = a - 7$, $\tilde{b} = b + 5$

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