MATH 3002. INTRODUCTION TO
MATHEMATICAL REASONING.
FALL 2015.
QUIZ 7

1. (30 points) 1) Give Peano's axiomatic of Arithmetic.

2) Since 1 is a special subject, the predicate $x = 1$ is well-defined. Give the predicative definition of $n = 3$.(You need to use the basic predicate $\sigma(m, n)$ - the natural number n is the successor of m).

Prove that the object "3" exists and unique.

1) $\mathbb{N}$.

- Special element $1 \in \mathbb{N}$
- Special predicative $\sigma(x,y)$ - (y - successor of x)

(i) $1 \in \mathbb{N}$;

(ii) $(\forall x)(\exists ! \, y)\, \sigma(x,y)$;

(iii) $(\forall x)(\forall y)(\forall z)\big(\sigma(x,z) \wedge \sigma(y,z)\big) \Rightarrow x = y$

(iv) $\sim(\exists x)\,\sigma(x,1)$; $(\forall x)(\exists y)\,\sigma(x,y) \Rightarrow y \neq 1$.

(v) $\{P(1) \wedge (\forall x)(\forall y)\,(P(x) \wedge \sigma(x,y)) \Rightarrow P(y)\} \Rightarrow (\forall z)\,P(z)$

2) $n = $ "2" $\Leftrightarrow \sigma(1,n)$; $(\exists n!)\; n = 2$.

$n = $ "3" $\Leftrightarrow \sigma(2, n)$;

or $n = $ "3" $\Leftrightarrow (\exists m)\,\sigma(1,m) \wedge \sigma(m,n)$.

From (ii), (iii) $n = $ "3" exists and unique.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. (70 points) Proce by the methos of mathematical induction that

$$1)S_n = \sum_{1 \leqq j \leqq n} j(j+1) = 1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = n(n+1)(n+2)/3;$$

$$2)2^n < n!, n \geqq 4;$$

$$3)n^2 < n!, n \geqq 4.$$

1. (i) Base: $k=1$. $2=2$. True.

(ii) Inductive conditional $P \Rightarrow Q$:

$S_k = k(k+1)(k+2)/3 \Rightarrow S_{k+1} = (k+1)(k+2)(k+3)/3$.

$P \Rightarrow S_{k+1} = S_k + (k+1)(k+2) = k(k+1)(k+2)/3 +$
$+ (k+1)(k+2) = (k+1)(k+2)\left(\frac{k}{3}+1\right) =$
$= \frac{(k+1)(k+2)(k+3)}{3}$.

True for $(\forall n)$.

2. (70 points) Proce by the methos of mathematical induction that

$$1) S_n = \sum_{1 \leqq j \leqq n} j \cdot j! = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1;$$

$$2) 2^n < n!, n \geqq 4;$$

$$3) n^2 < n!, n \geqq 4.$$

1). $S_1 = 1$ (base of induction);

$S_k = (k+1)! - 1 \Rightarrow S_{k+1} = (k+2)! - 1$ (inductive conditional)

$P \Rightarrow Q$

Let $P = (S_k = (k+1)! - 1) \Rightarrow S_{k+1} = (k+1)! - 1 + (k+1)\cdot(k+1)!$

$= (k+1)!\,(1 + (k+1)) - 1 = (k+1)!\,(k+2) - 1 = (k+2)! - 1 \Leftrightarrow Q.$

2) $2^n < n!$, $n \geq 4$.

P $n = 4 \Rightarrow 2^4 < 4!$; $16 < 24$. True (Base)

Inductive conditional:

$$((2^k < k!) \wedge k \geq 4) \Rightarrow (2^{k+1} < (k+1)!).$$

$$P \Rightarrow Q$$

$P \Rightarrow (k \geq 4 \Rightarrow 2^{k+1} < 2k! \Rightarrow$

$2^{k+1} < (k+1)!$ (Since $k+1 > 2$) $\Rightarrow Q$.

True for $(\forall n)\ n \geq 4$.

3) $n^2 < n!$, $n \leq 4$.

Equivalent $\underline{n < (n-1)!}$

1. $n=4$. $4 < 3!$: $4 < 6$. True

2. $P \Rightarrow Q$. Inductive conditional

$$\left(k \geq 4 \wedge k < (k-1)!\right) \overset{?}{\Rightarrow} k+1 < k! \Leftrightarrow$$

$$\Leftrightarrow 1 < k\left((k-1)! - 1\right) \cancel{\Leftarrow 4} \quad \text{True.}$$

$$k \geq 4 \wedge (k-1)! \geq 6 \quad (k \geq 4).$$

Another solution

$$k! > k^2 \Rightarrow (k+1)k! = (k+1)! > k^2(k+1) > (k+1)^2$$

Since $k^2 > k+1$ for $k \geq 2$.