MATH 3002. INTRODUCTION TO MATHEMATICAL REASONING.

FALL 2015.

QUIZ 6

1. (30 points) 1) Write as a predicative formula Theorem: For any pair of parallel lines there is a line intersecting them both.

2. Prove this Theorem using axioms which we stated.

1) $(\forall \ell)(\forall m)\ \ell \parallel m \Rightarrow (\exists n)(\exists x)(\exists y)\ x \in \ell \wedge x \in n \wedge y \in m \wedge y \in n.$

2) $(\forall x)\ x \in \ell \wedge (\forall y)\ y \in m \underset{(\text{axiom})}{\Rightarrow} (\exists!\, n)\ x \in n \wedge y \in n$

$\Rightarrow (x \in m \wedge x \in n) \wedge (y \in \ell \wedge y \in n).$

2. (40 points)

1) Give the convenient negation of the predicate

$$(\forall x)(\exists! y)((\ P(x,y) \vee S(x)) \Longrightarrow ((\forall z)(Q(z) \Leftrightarrow P(z))).$$

2) Prove that the formula

$$(\exists x)(\forall y)P(x,y) \Rightarrow (\forall y)(\exists x)P(x,y);$$

is tautology but

$$(\exists x)(\forall y)P(x,y) \Leftarrow (\forall y)(\exists x)P(x,y)$$

is not.

1) $(\forall x)(\exists! y)\left(\left(P(x,y) \vee S(x)\right) \wedge \sim\left((\forall z)(Q(z) \Leftrightarrow P(z)\right)\right)$

$\Leftrightarrow (\forall x)(\exists ! y)\left(\left(P(x,y) \vee S(x)\right) \wedge (\exists z)\left((Q(z) \wedge \sim P(z)) \vee\right.\right.$

$\left.\left.(\sim Q(z) \vee P(z)\right)\right).$

2. Denote $A \Rightarrow B$, $A \Leftarrow B$.

A - True: For some x_0 $(\forall y)(P(x_0,y))$

B - True: There is $x(y)$ s.t. $(\forall y) P(x(y),y))$.

$A \Rightarrow B$ - Take $x(y) \equiv x_0$;

$B \Rightarrow A$ - Can be False: $x(y)$ exist but no constant x_0.

Example: $P(x,y)$ - $x<y$ $[\mathbb{R}]$. We can take $x(y) = y-1$ but no x_0 s.t. $x_0 < y$ $\forall y$.

3. (30 points) Prove in the predicative form for natural numbers that mn has a form $3k+1$ iff m and n don't multiple 3 and have the same remainders in the division on 3.

$$(\forall m)(\forall n)(\exists k)\ mn = 3k+1 \Leftrightarrow 3 \nmid n \wedge 3 \nmid m \wedge (\exists r)(\exists q_1)(\exists q_2)\ 0 < r < 3 \wedge$$

$$m = 3q_1 + r \wedge n = 3q_2 + r.$$

$$a \Leftrightarrow B$$

$r = 1 \wedge r = 2.$

$B \Rightarrow a$

$$B \Rightarrow (m = 3q_1 + 1 \wedge n = 3q_2 + 1) \vee$$

$$\vee (m = 3q_1 + 2 \wedge n = 3q_2 + 2) \Rightarrow$$

$$\Rightarrow (\exists k)\ mn = 3k + 1$$

$a \Rightarrow B$ Contraposition

$$\sim B \Rightarrow (3 \mid n \wedge 3 \mid m) \vee (m = 3q_1 + 1 \wedge n = 3q_2 + 2) \Rightarrow$$

$$(3 \mid mn) \vee (\exists k)\ mn = 3k + 2 \Rightarrow \sim a.$$