MATH 3002. INTRODUCTION TO
MATHEMATICAL REASONING.
FALL 2015.
QUIZ 5

1. (30 points) 1) Using the predicate "a point x lies on a line l" give a predicative formula for

-the definition of parallel lines;

-the postulate that through a point outside a line is passing a parallel line and this parallel is unique.

2) Write down a predicative formula for

"if a solution of an equation exists it is unique. "

Give the convenient (useful) negation of this predicate.

[X-points]

1) $x \in l$: Point x lies on the line l. [L-lines]

a) $l \parallel m \overset{def}{\Leftrightarrow} \sim ((\exists x)\ x \in l \wedge x \in m)$ | No joint points

$\Leftrightarrow (\forall x)\ x \notin l \vee x \notin m.$

(it could be $x \notin l \wedge x \notin m$.

b) $(\forall x)(\forall l)\ x \notin l \Rightarrow (\exists m!)\ x \in m \wedge m \parallel l.$

2) [$\mathbb{R}$], [F - equations]

$(\exists x)\ F(x) = 0 \Rightarrow \left(\left((\forall y)\left(F(y) = 0\right) \Rightarrow y = x\right)\right)$

Negation

$(\exists x)\ F(x) = 0 \wedge \left(\left(\exists y\right) F(y) = 0 \wedge y \neq x\right) \Leftrightarrow$

$(\exists x)(\exists y)\ F(x) = 0 \wedge F(y) \wedge x \neq y$

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2. (40 points) 1. Give convenient negation of the conditional and the biconditional.

2. Give the convenient negation of the predicate

$$(\forall x)(\exists y)(P(x,y) \Longrightarrow (\exists z)(Q(z) \vee P(x))).$$

1) $\sim(P \Rightarrow Q) \Leftrightarrow \sim(\sim P \vee Q) \Leftrightarrow \underline{P \wedge \sim Q}$

$\sim(P \Leftrightarrow Q) \Leftrightarrow \sim((P \wedge Q) \vee (\sim P \wedge \sim(Q)))$

$\Leftrightarrow \sim(P \wedge Q) \wedge \sim(\sim P \wedge \sim Q)$ de Morgen

$\Leftrightarrow (\sim P \vee \sim Q) \wedge (P \vee Q)$ double negation

$\Leftrightarrow ((\sim P \vee \sim Q) \wedge P) \vee ((\sim P \vee \sim Q) \wedge Q)$

$\Leftrightarrow \underline{(\sim Q \wedge P) \vee (Q \wedge \sim P)}$ $(P \wedge \sim P \Leftrightarrow F$, $R \vee F \Leftrightarrow R$

2) $\cancel{(\exists x)(\forall y) \sim P(x,y)}$ $\sim((\forall x)(\forall y)P(x,y) \Rightarrow (\exists z)(Q(z) \vee P(x)$

$\cancel{(\forall x)(\exists y)P(x,y) \wedge ((\forall z) \sim (Q(z) \wedge P(x)))}$

$\cancel{\Leftrightarrow (\forall x)(\exists y)P(x,y) \wedge ((\forall z)(\sim Q(z) \vee \sim P(x))}$

$\sim((\forall x)(\exists y)P(x,y) \Rightarrow (\exists z)(Q(z) \vee P(x))) \Leftrightarrow$

$(\exists x)(\forall y)[P(x,y) \wedge (\forall z)(\sim Q(z) \wedge \sim P(x))] \Leftrightarrow$

$(\exists x)(\forall y)\cancel{P(x,y)}(\forall z)[P(x,y) \wedge \sim Q(z) \wedge \sim P(x)].$

3. (30 points) Prove in the predicative form for natural numbers:

1) $3|n^2 \vee 3|n^2 - 1$;

2) if mn has a form $3k - 1$ then either m or n has such a form.

Lemma. $(\forall n)(3|n \vee (\exists k)\, n=3k+1 \vee (\exists l)\, n=3l-1$.

Follows from $(\forall n)(3|n \vee (\exists k)\, n=3k+1 \vee (\exists m)\, n=3k+2)$.

(Division with remainder)

1) $3|n \Rightarrow 3|n^2$; $n=3k+1 \Rightarrow n^2=3(3k^2+2k)+1 \Rightarrow 3|n^2-1$;

$n=3k-1 \Rightarrow n^2=3(3k^2-2k)+1 \Rightarrow 3|n^2-1$.

2) $P: (\exists k)\, mn=3k-1$; $Q: (\exists r)\, m=3r-1 \vee (\exists s)\, n=3s-1$

$P \Rightarrow Q$?

Contraposition: $\sim Q \Rightarrow \sim P$

$\sim Q: 3|m \vee 3|n \vee ((\exists a)\, m=3a+1 \wedge (\exists b)\, n=3b+1) \Rightarrow$

$3|mn \vee \overset{(\exists c)}{(mn=(3a+1)(3b+1)=3c+1}$

Contradiction.