MATH 300. INTRODUCTION TO MATHEMATICAL REASONING.
FALL 2015.
MIDTERM 2

1. (25 points) Let us consider the geometrical sequence with the recursive definition $a_{k+1} = qa_k$.

1) Give the formula for a_n through a_1 and q and prove it by the method of mathematical induction.

2) By the same method to prove the formula for the sum of n terms:

$$S_n = a_1 + \cdots + a_n + \cdots = \frac{a_1(q^n - 1)}{q - 1}.$$

1) $a_n = a_1 q^{n-1}$.

(i) $a_1 = a_1$.

(ii) $a_k = a_1 q^{k-1} \overset{?}{\Rightarrow} a_{k+1} = a_1 q^k$

$a_k = a_1 q^{k-1} \Rightarrow a_{k+1} = a_k q = a_1 q^k$. True

2) (i) $S_1 = a_1$

(ii) $S_k = \dfrac{a_1(q^k - 1)}{q-1} \overset{?}{\Rightarrow} S_{k+1} = \dfrac{a_1(q^{k+1}-1)}{q-1}$

$P \Rightarrow S_{k+1} = \dfrac{a_1(q^{k} - 1)}{q-1} + \cancel{q}\, a_1 q^k = \dfrac{a_1(q^{k}-1) + a_1 q^k (q-1)}{q-1}$

$= \dfrac{a_1(q^{k+1} - 1)}{q-1}$

2. (25 points) Prove that

$$S_n = 1 + 3^3 + 5^3 \cdots + (2n-1)^3 = n^2(2n^2-1).$$

(i) $n=1 \Rightarrow 1=1;$ True

(ii) $S_k = k^2(2k^2-1) \Rightarrow S_{k+1} =$

$$= (k+1)^2(2(k+1)^2-1);$$

$$= (k^2+2k+1)(2k^2+4k+1) = 2k^4+8k^3+11k^2+6k+1$$

$$S_k \Rightarrow S_{k+1} = k^2(2k^2-1) + \cancel{(2(k+1))^3} + (2k+1)^3 =$$

$$= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 =$$

$$= 2k^4 + 8k^3 + 11k^2 + 6k + 1.$$

3. (25 points) Prove that

$$24|(n^2-1)n(n+2)$$

by two methods:
directly, using properties of divisibility and
by the method of mathematical induction.

1) Direct. $(n^2-1)n(n+2)=(n-1)n(n+1)(n+2)$.

The product of 4 consequent natural numbers.
- at least 1 is multiple 3;
- " 2 consequent even (one multiple 4).

2) Mathematical induction.

(i) $n=1$. $24|0$ True

(ii) $24|\underbrace{(k^2-1)k(k+2)}_{a=} \Rightarrow 24|\underbrace{k(k+1)(k+2)(k+3)}_{b=}$
$=(k-1)k(k+1)(k+2)$

$b-a=k(k+1)(k+2)[k+3-k+1]=4k(k+1)(k+2)$

Prove that $6|c=k(k+1)(k+2)$
- at least one factor multiple 3
- — " — " — is even

$6|b-a$ ~~&~~ $24|a \Rightarrow 24|b$

4. (25 points) Show that any exact amount of postage that is an integer number of cents greater than 37 cents can be formed using just 8-cent and 5-cent stamps. (Prove by induction).

Basic relations:

$$8\cdot 2 - 5\cdot 3 = 1$$
$$5\cdot 5 - 8\cdot 3 = 1$$

~~n=38~~ $n = 8a + 5b, \quad a \geq 0, b \geq 0.$

(i) $n = 38, \quad 38 = 8 + 5\cdot 6, \; a = 1, b = 6$

(ii) $k = 8a + 5b \wedge n \geq 38 \Rightarrow (\exists \tilde{a})(\exists \tilde{b})\; k+1 = 8\tilde{a} + 5\tilde{b}$

1) $b \geq 3 \Rightarrow \tilde{b} = b - 3 \wedge \tilde{a} = a + 2;$

2) $b \leq 2 \Rightarrow 8a \geq 28 \Rightarrow a \geq 4 \Rightarrow \tilde{a} = a - 3 \wedge \tilde{b} = b + 5$

5. (25 points) Prove that the formula

$$(\exists x)(\forall y)P(x,y) \Rightarrow (\forall y)(\exists x)P(x,y);$$

is tautology but

$$(\exists x)(\forall y)P(x,y) \Leftarrow (\forall y)(\exists x)P(x,y)$$

is not.

1) $a \Rightarrow B$. Let $(\forall y) P(x_0, y)$ is True.
$a \Rightarrow \Uparrow$ for some x_0.

Then take at B $x(y) \equiv x_0$ and B is True.

2) Let $P(x,y) = \{x+y=0\}$ $\mathbb{R}$

B is True: $x(y) = -y$.

a is False.

6. (25 points)1) Write down as a predicative formula with quantifiers only on the set of All real numbers the statement:

For each nonnegative real number x exists and unique such y that $y^2 = x$.

2) Write down in the convenient form negation of this statement and prove it (at predicative form).

1) $(\forall x)\ x \geq 0 \Rightarrow (\exists! y)\ y^2 = x$;

2) $(\exists x)\ x \geq 0 \wedge [(\forall y)\ y^2 \neq x \vee (\exists u)(\exists v)\ u^2 = x \wedge v^2 = x \wedge u \neq v]$.

2) is True: Take any $x \geq 0$; let $x = 4$, $u = 2$, $v = -2$

Then 2nd term of the disjunction is true!

$(2)^2 = 4 \wedge (-2)^2 = 4 \wedge 2 \neq -2$.

7. (25 points) 1)State Peano's axioms of Arithmetic.

2) Using directly by Peano axioms give the definition of the function on natural numbers $f(n) = 2n$.

1) $\mathbb{N}$

$1 \in \mathbb{N}$ - special element

$\sigma(x, y)$ - y - successor of x.

(i) $1 \in \mathbb{N}$

(ii) $(\exists x)\ (\exists!\, y)\ \sigma(x,y)$;

(iii) $(\forall x)(\forall y)(\forall z)\big(\sigma(x,z) \vee \sigma(y,z)\big) \Rightarrow x = y$;

(iv) $(\forall x)\ (\exists y)\ \sigma(x,y) \Rightarrow y \neq 1$;

(v) $\{P(1) \wedge (\forall x)(\forall y)\ (P(x) \wedge \sigma(x,y)) \Rightarrow P(y)\} \Rightarrow$

$\Rightarrow (\forall z)\ P(z)$.

2) (i) $\cancel{f(1) \Rightarrow}$ $\sigma(1, x) \Rightarrow f(1) = x$; on (ii) $(\exists!)\, x$

(ii) $f(k) = (x \wedge \sigma(x,y) \wedge \sigma(y,z)) \Rightarrow \cancel{f(} \wedge \sigma(k, k+1)$

$\Rightarrow f(k+1) = z$,

$y = k+1 \Leftrightarrow \sigma(k, y)$.

8. (25 points) 1) Define in the predicative form that $\lim_{n\to\infty} a_n = L$.
2) Give the convenient negation of this definition.
3) Definite that a sequence $\{a_n\}$ divergences (has no limit).
4) Find $\lim_{n\to\infty} 2/n$ and prove it on the language of ε, N.

1) $\forall \varepsilon \left[\varepsilon > 0 \Rightarrow \left((\exists N)(\forall n) \left[n > N \Rightarrow |L - a_n| < \varepsilon \right] \right) \right]$

2) $\exists \varepsilon \left[\varepsilon > 0 \wedge \left((\forall N)(\exists n) \left[n > N \wedge |L - a_n| \geq \varepsilon \right] \right) \right]$.

3) $\forall L \{ - \quad - \quad - \quad - \quad - \}$

4) $\lim\limits_{n\to\infty} \frac{2}{n} = 0.$

Proof. $\frac{2}{n} < \varepsilon \Leftrightarrow \frac{2}{\varepsilon} < n$. Let $N(\varepsilon) = [\frac{2}{\varepsilon})$.

$(\forall n)\ n > N(\varepsilon) \Rightarrow \frac{2}{n} = |0 - \frac{2}{n}| < 0.$