MATH 300. INTRODUCTION TO MATHEMATICAL REASONING. FALL 2015. MIDTERM 1

1. (25 points) Prove the equivalence

$$(P \vee Q) \wedge (R \vee S) \Leftrightarrow (P \wedge R) \vee (P \wedge S) \vee (Q \wedge R) \vee (Q \wedge S)$$

without using True tables.

$$(P \vee Q) \wedge (R \vee S) \Leftrightarrow ((P \vee Q) \wedge R) \vee ((P \vee Q) \wedge S) \quad \text{Distr. for } \wedge$$

$$\Leftrightarrow ((P \wedge R) \vee (Q \wedge R)) \vee ((P \wedge S) \vee (Q \wedge S)) \quad \text{Distr. for } \wedge$$

$$\Leftrightarrow (P \wedge R) \vee (P \wedge S) \vee (Q \wedge R) \vee (Q \wedge S)$$

Commut. and Associative for $\vee$.

2. (25 points) Transform to the convenient form(negations can be only on variables)

$$\sim(\sim(P\vee\sim(S\wedge\sim R))\wedge(P\vee\sim S)).$$

$\Leftrightarrow (P \vee \sim(S \wedge \sim R)) \vee \sim(P \vee \sim S)$ de Morgan, double negation

$\Leftrightarrow (P \vee \sim S \vee R) \vee (\sim P \wedge S)$ de Morgan double negation associativity

$\Leftrightarrow (P \vee \sim S \vee R \vee (\sim P \wedge S))$ absorption

$\Leftrightarrow \sim S \vee R \vee (P \vee S) \Leftrightarrow$

~~associative~~

$\cancel{\Leftrightarrow R \vee (\sim S \vee P)}$

$\cancel{\Leftrightarrow R \vee \sim S \vee P}$. associativity

$\Leftrightarrow \sim S \vee S \vee R \vee P \Leftrightarrow T$

$(\sim S \vee S) \equiv T$

$T \vee q \equiv T$

We could see it earlier!

$\underbrace{(P \vee \sim S)}_{q} \vee S \vee \underbrace{\sim (P \vee \sim S)}_{\sim q}$ $\quad q \vee \sim q \equiv T$

3. (25 points) a) The propositional function $f(P_1, P_2, P_3, P_4)$ is True iff exactly 2 variables are True. Construct its True table and represent it by a formula using conjuctions, disjunctions, negations.

b) Using only disjunctions and negations.

P_1	P_2	P_3	P_4	f
T	T	T	T	F
T	T	T	F	F
T	T	F	T	F
T	T	F	F	T
T	F	T	T	F
T	F	T	F	T
T	F	F	T	T
T	F	F	F	F
F	T	T	T	F
F	T	T	F	T
F	T	F	T	T
F	T	F	F	F
F	F	T	T	T
F	F	T	F	F
F	F	F	T	F
F	F	F	F	F

16 lines

$\binom{4}{2} = 6$ lines with

$f = T$

$$f(P_1, P_2, P_3, P_4) \Leftrightarrow (P_1 \wedge P_2 \wedge \sim P_3 \wedge \sim P_4) \vee (P_1 \wedge \sim P_2 \wedge P_3 \wedge \sim P_4) \vee$$

$$\vee (P_1 \wedge \sim P_2 \wedge \sim P_3 \wedge P_4) \vee (\sim P_1 \wedge P_2 \wedge P_3 \wedge \sim P_4) \vee$$

$$\vee (\sim P_1 \wedge P_2 \wedge \sim P_3 \wedge P_4) \vee (\sim P_1 \wedge \sim P_2 \wedge P_3 \wedge P_4).$$

b) We use $A \wedge B = \sim(\sim A \vee \sim B)$

$$f \Leftrightarrow \sim(\sim P_1 \vee \sim P_2 \vee P_3 \vee P_4) \vee \sim(\sim P_1 \vee P_2 \vee \sim P_3 \vee P_4) \vee$$

$$\vee \sim(\sim P_1 \vee P_2 \vee P_3 \vee \sim P_4) \vee \sim(P_1 \vee \sim P_2 \vee \sim P_3 \vee P_4)$$

$$\vee \sim(P_1 \vee \sim P_2 \vee P_3 \vee \sim P_4) \vee \sim(P_1 \vee P_2 \vee \sim P_3 \vee \sim P_4).$$

another solution of b): $((a \vee \sim B) \Rightarrow B) \Leftrightarrow (\sim a \wedge B) \vee B \Leftrightarrow (\sim a \vee B) \wedge B$

$\Leftrightarrow B \not\Leftrightarrow a \Rightarrow B$

It's T if B - T; if B - F, a - T, its F.

4. (25 points) Determine whether each of the following is a tautology, contradiction, or neither. Don't use True tables but transform formulas

$$((A \wedge \sim B) \Rightarrow \sim A) \Leftrightarrow (\sim B \Rightarrow \sim A);$$

$$((A \vee \sim B) \Rightarrow B) \Leftrightarrow (A \Rightarrow B);$$

$$A \wedge \mathbf{F}.$$

a) $(P \Rightarrow Q) \Leftrightarrow (\sim P \vee Q)$.

$((a \wedge \sim B) \Rightarrow \sim a) \Leftrightarrow \sim (a \wedge \sim B) \vee \sim a$

$\Leftrightarrow (\sim a \vee B) \vee \sim a \Leftrightarrow \sim a \vee B \Leftrightarrow B \vee \sim a \Leftrightarrow$

de Morgan

Commutativ

$\Leftrightarrow \sim B \Rightarrow \sim a$. Tautology

b) Let $\underline{a - F, B - F}$. Then $\underline{a \Rightarrow B \text{ is } T}$

$a \vee \sim B - T$, $\underline{(a \vee \sim B) \Rightarrow B - F}$

If B is False we have True on both sides. So it's neither

If a -

c) $a \wedge F \Leftrightarrow F$ Contradiction

5. (25 points) Prove that the product of 2 natural numbers is odd iff they both are odd.

P - $m = 2k+1 \wedge n = 2l+1$

Q - $mn = 2s+1$.

$P \Leftrightarrow Q$

1) $P \Rightarrow Q$ $\quad mn = (2k+1)(2l+1) =$

$= 2(2kl+k+l)+1$

$= 2s+1 \Rightarrow Q$

$s = 2kl+k+l$

2. $Q \Rightarrow P$. By contradiction:

$\sim P \Rightarrow \sim Q$. $\sim P$ - "at least one of m, n is even.

$\sim P$ - Let m - is even; $m = 2u$.

Then $mn = 2(un)$ - even - contradicition $\sim Q$

6. (25 points) Prove that the set of prime numbers is infinite.

By a reduction to Contradiction:

Suppose that there just a finite number of prime numbers:

$p_1 < p_2 < \dots < p_k$

and let $N = \cancel{p_1 \dots} \; p_1 p_2 \cdots p_k + 1$.

Then - N can't be prime since

$\cancel{p} \; n > p_k$ (biggest prime number)

- N can't be composite since

$\forall i \quad p_i | N-1 = p_1 \cdots p_k$

and consequent numbers $(N-1, N)$ can't have joint factors $\neq 1$. (Either for all p_j remainder of N is 1.

Contradiction

7. (25 points) Prove by 2 ways (by direct proof and by the contraposition that

$$x^2 - 3x + 2 < 0 \Leftrightarrow 1 < x < 2. \qquad P \Leftrightarrow Q$$

$$x^2 - 3x + 2 = (x-2)(\cancel{2}\,x-1)$$

$$P \Leftrightarrow (x-2)(x-1) < 0$$

Proof. $P \Rightarrow Q$

$$(x-2)(x-1) < 0 \Rightarrow ((x-2)>0 \wedge (x-1)<0) \vee$$
$$\vee ((x-2)<0 \wedge (x-1)>0) \Leftrightarrow (x>2 \wedge x<1)$$
$$\vee (x<2 \wedge x>1) \Leftrightarrow F \vee 1<x<2 \Leftrightarrow Q$$

$Q \Rightarrow P$

$$1<x<2 \Leftrightarrow x>1 \wedge x<2 \Leftrightarrow (x-1)>0 \wedge (x-2)<0 \Rightarrow$$
$$\Rightarrow (x-1)(x-2)<0 \Leftrightarrow P.$$

$P \Rightarrow Q$. By contradiction: $\sim Q = x \leq 1 \vee x \geq 2$

2cases: if $x \geq 2 \Rightarrow (x-1 \geq 1 > 0) \wedge (x-2 \geq 0) \Rightarrow$
$$\Rightarrow (x-2)(x-1) \geq 0 \Leftrightarrow \sim P$$
$$x \leq 1 \Rightarrow ((x-1) \leq 0 \wedge (x-2) \leq -1 < 0) \Rightarrow (x-2)(x-1) \geq 0 \Leftrightarrow \sim P.$$

$Q \Rightarrow P$. By contradiction: $\sim P \Rightarrow \sim Q$

$$\sim P \Leftrightarrow x^2 - 3x + 2 = (x-2)(x-1) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \big((x-2) \geq 0 \wedge (x-1) \geq 0\big) \vee \big((x-2) \leq 0 \wedge (x-1) \leq 0\big)$$

$$\Leftrightarrow (x \geq 2 \wedge x \geq 1) \vee (x \leq 2 \wedge x \leq 1)$$

$$\Leftrightarrow \cancel{(x \geq 2)}\ x \geq 2 \vee x \leq 1 \Leftrightarrow \sim P.$$

8. (25 points) Prove that n is even iff $8|n^2 - 2n$.

$P \Leftrightarrow Q$; P – n-even; Q – $8 | n^2-2n$

$P \Rightarrow Q$ $\quad n^2 - n = n(n-2) \Rightarrow n, n-2$ – consequitive even numbers.

$\exists r \quad n = 2r, \; n-2 = 2r-2 = 2(r-1)$.

$r \vee r-1$ is even $\Rightarrow$

one of $n, n-2$ is multiple 4. $\Rightarrow 8 | n^2 - n$.

$Q \Rightarrow P$ ~~Comp~~ ~~Contradiction~~

~~$\sim P \Rightarrow \sim Q$~~

~~$\sim P$~~

$Q \Rightarrow P$, By contraposition: $\sim P \Rightarrow \sim Q$

$\sim P$: n - odd $\Leftrightarrow n = 2k+1 \Rightarrow n-2 = 2k-1 \Rightarrow n(n-2) = (2k+1)(2k-1)$

$\Rightarrow n^2 - 2n$ is odd $\Rightarrow \sim Q$.