MATH 300.2. INTRODUCTION TO
MATHEMATICAL REASONING.
FALL 2015.
FINAL EXAM

1. (30 points) Prove by the method of mathematical induction:

$$1 \cdot 2^2 + 2 \cdot 3^2 + \cdots + n(n+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5).$$

$n = 1$ $\qquad 4 = \frac{36}{12}$

Inductive conditional:

$S_n = \frac{1}{12} n(n+1)(n+2)(3n+5) \Rightarrow$

$S_{n+1} = \frac{1}{12} (n+1)(n+2)(n+3)(3n+8)$

$S_n = \ldots \Rightarrow S_{n+1} = \frac{1}{12}(n+1)(n+2)\left[n(3n+5) + 12(n+2)\right]$

$= \frac{1}{12}(n+1)(n+2)\left[3n^2 + 17n + 24\right]$

$(n+3)(3n+8) = 3n^2 + 17n + 8.$ True

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. (30 points) Let us consider the sequence with the recursive definition $a_1 = 1, a_{k+1} = a_k + 2$.

1) Give the formula for a_n and prove it by the method of mathematical induction.

2) By the same method to prove the formula for the sum of n terms:

$$S_n = a_1 + \cdots + a_n = n^2.$$

1. $a_n = 2n - 1$

$a_1 = 1$

$a_k = 2k - 1 \Rightarrow a_{k+1} = a_k + 2 = 2k + 1 = 2(k+1)$

2. $S_k = k^2 \Rightarrow S_{k+1} = (k+1)^2$

$S_k = k^2 \Rightarrow S_{k+1} = k^2 + (2k+1) = (k+1)^2$

3. (30 points) Prove the tautology

$$((P \vee Q) \wedge (\sim P \vee \sim Q)) \Leftrightarrow ((P \wedge \sim Q) \vee (\sim P \wedge Q)).$$

by 2 methods:
-using True tables;
- using Propositional Laws without Tables.

$a \Leftrightarrow B$

P	Q	a	B	$a \Leftrightarrow B$
T	T	F	F	T
F	F	F	F	T
F	T	T	T	T
T	F	T	T	T

distr.

2) $a \Leftrightarrow (P \wedge (\sim P \vee \sim Q)) \vee (Q \wedge (\sim P \vee \sim Q)) \Leftrightarrow$

$(F \vee (P \vee \sim Q)) \vee ((Q \wedge \sim P) \vee \sim Q) \Leftrightarrow B$

$P \wedge \sim P = F$ distr. ~~dissoc.~~ comm.

4. (30 points) 1.Transform to the convenient form the formula.

$$\sim [(P \Leftrightarrow \sim (Q \wedge \sim R)) \Leftarrow (\sim S \Rightarrow T)].$$

Simplify the answer.

2. Transform it in a formula with only disjunctions and negations.

$$\sim (a \Leftrightarrow b) \Leftrightarrow (\sim a \vee b) \wedge (a \vee \sim b)$$

$$\sim (a \Rightarrow b) \Leftrightarrow a \wedge \sim b.$$

$$(\sim S \Rightarrow T) \wedge \sim (P \Leftrightarrow \sim (Q \wedge \sim R)) \Leftrightarrow$$

$$(\sim S \Rightarrow T) \wedge [(\sim P \vee \sim (Q \wedge \sim R)) \wedge (P \vee (Q \wedge \sim R))]$$

$$(S \vee T) \wedge [(\sim P \vee \sim Q \vee R) \wedge (P \vee (Q \wedge \sim R))]$$

$$\sim [\sim (S \vee T) \vee \sim [\sim (\sim P \vee \sim Q \vee R) \wedge (P \vee (Q \wedge \sim R))$$

$$\Leftrightarrow \sim [\sim (S \vee T) \vee (\sim P \vee \sim Q \vee R) \vee \sim (P \vee \sim (\sim Q$$

5. (30 points) Determine whether each of the following is Tautology, Contradiction, or neither. Don't use True Tables.

$$((P \vee \sim Q) \Rightarrow P) \Leftrightarrow (P \Rightarrow Q);$$

$$(P \Rightarrow (Q \vee R)) \Leftrightarrow ((P \wedge \sim R) \Rightarrow Q);$$

$$\mathbf{F} \wedge P.$$

1) $A \Leftrightarrow$

$A \Leftrightarrow \sim(P \vee \sim Q) \vee P \Leftrightarrow (\sim P \wedge Q) \vee P \Leftrightarrow$

$(\sim P \vee P) \wedge (Q \vee P) \Leftrightarrow T \wedge (Q \wedge P) \Leftrightarrow Q \wedge P$

$Q \wedge P \not\Leftrightarrow P \Rightarrow Q$

$P = F,\ Q = T$ | $P = T,\ Q = F$

$P \Rightarrow Q = T$ | $= F$

$Q \wedge P = F.$ | $= F$

neither

2) $A \Leftrightarrow \sim P \vee (Q \vee R) = \sim P \vee Q \vee R$

$B \Leftrightarrow \sim(P \wedge \sim R) \vee Q \Leftrightarrow \sim P \vee R \vee Q$ Tautology

3) $F \wedge P \Leftrightarrow F$ negative.

6. (30 points) 1. Prove the tautology

$$(\exists x)(P(x) \vee Q(x)) \Leftrightarrow (\exists y)P(y) \vee (\exists z)Q(z);$$

2. Express the unique existential quantifier through other quantifiers and construct its convenient negation.

1. $A \Leftrightarrow B$

A is True iff ~~∃~~ for some x_0

$P(x_0) \vee Q(x_0)$ is True. So $P(x_0) = T$ or

$Q(x_0) = T$.

If $P(x_0) = T$ then $(\exists y) P(y)$;

if $Q(x_0) = T$ then $(\exists z) Q(z)$.

~~&~~ So in the both cases B is True.

2. $(\exists! x) A(x) \Leftrightarrow (\exists x) A(x) \wedge (\forall y)(\forall z)[(A(y) \wedge A(z)) \Rightarrow y = z]$

$\sim(\exists! x) A(x) \Leftrightarrow (\forall x) \sim A(x) \vee (\exists y)(\exists z)[A(y) \wedge A(z) \wedge y \neq z]$

7. (30 points) State using predicates and quantifiers and prove on this language that n^2+1 for natural n is never multiple 3.

$$(\forall n)\ 3 \nmid (n^2+1) \qquad (\mathbb{N}).$$

Lemma. $(\forall n)\ 3 \mid n \vee (\exists k)\ n = 3k+1 \vee (\exists l)\ n = 3l-1$

Corollary. $(\forall n)\ 3 \mid n^2 \vee (\exists m)\ n^2 = 3m+1$.

$3 \mid n \Rightarrow 3 \mid n^2$

$(3k+1)^2 = 9k^2+6k+1 \Rightarrow (\exists m)\ n^2 = 3m+1$

$m = 3k^2+2k$.

Proof. $3 \mid n^2 \vee n^2 = 3m+1 \Rightarrow n^2+1 = 3l+1 \vee n^2+1 = 3l+2$

8. (30 points) For which x we have $x^2 - 3x - 10 > 0$; give the complete description. Use predicates at Proof.

$$x^2 - 3x - 10 = (x-5)(x+2)$$

$$x^2 - 3x - 10 > 0 \Leftrightarrow \cancel{x < 5 \vee x <} \; x < 2 \vee x > 5.$$

Proof

$$(x^2 - 3x - 10) > 0 \Leftrightarrow (x-5)(x+2) > 0 \Leftrightarrow$$

$$\Leftrightarrow (x-5 > 0 \wedge x+2 > 0) \vee (x-5 < 0 \wedge x+2 < 0)$$

$$\Leftrightarrow (x > 5 \wedge x > -2) \vee (x < 5 \wedge x < -2).$$

$$\Leftrightarrow x > 5 \vee x < -2.$$

Result: $(x^2 - 3x - 10) \Leftrightarrow x > 5 \vee x < -2.$

9. (30 points) 1. Give predicative definitions what does mean that a sequence is

-bounded;

-convergent.

2. Prove that any convergent sequence is bounded.

1. $(\exists M)(\forall n)\ |x_n| < M$

$(\exists L)(\forall \varepsilon)\ \varepsilon > 0 \Rightarrow \left[(\exists N)(\forall n)\ n > N \Rightarrow |x_n - L| < \varepsilon\right.$

2.

$(\exists L) \ldots$

Take L and $\varepsilon = 1$. Then

$L - 1 < x_n < L + 1$ for $n > N = N(1)$.

Finite number x_n, $< N$, bounded!

$(\exists P)\ |x_n| < P$.

Take $M = \max(L, P)$.

10. (30 points) 1) Give the predicative definition that a sequence is
-unbounded;
-divergent.
2) Find limits if $n \to \infty$ of the following sequences, if they exist, and prove the existence or non existence, using directly the definitions:

$$\frac{n^2-1}{n^2+1}; \frac{3^n+2^n}{2^{2n}+\sin^2 x}, 2^n.$$

Unbounded:

$(\forall M)(\exists n)\ |x_n| \geq M$

Divergent:

$(\forall L)(\exists \varepsilon)\ \varepsilon > 0 \wedge [(\forall N)(\exists n)\ n > N \wedge |x_n - L| \geq \varepsilon]$

2. 1) $L = 1$

$|x_n - L| = \left|\frac{n^2-1}{n^2+1} - 1\right| = \frac{2}{n^2+1} < \frac{2}{n} < \varepsilon, \quad N(\varepsilon) = \left[\frac{2}{\varepsilon}\right]$

2) $L = 0$

$\left|\frac{3^n+2^n}{2^{2n}+\sin^2 x}\right| < \frac{2 \cdot 3^n}{4^n} < \varepsilon \Leftrightarrow \left(\frac{3}{4}\right)^n < \frac{\varepsilon}{2}$

$\left(\frac{3}{4}\right)^n < \frac{\varepsilon}{2} \Leftrightarrow n(\ln 3 - \ln 4) < \ln \varepsilon$

$n \ln 4 > n(\ln 4 - \ln 3) > -\ln \varepsilon \qquad N(\varepsilon) = \left[-\frac{\ln \varepsilon}{\ln 4}\right]$

2^n - divergent.

2^n - unbounded: $\forall M>0$, $n>\log_2 M \Rightarrow 2^n > M$.

$\lim 2^n = L \Rightarrow n > N(1) \Rightarrow 2^n < L-1$.