MATH 300.1. INTRODUCTION TO
MATHEMATICAL REASONING.
FALL 2015.
FINAL EXAM

1. (30 points) Prove by the method of mathematical induction:

$$S_n = 1 + \frac{1}{2^2} + \cdot + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Base: $n=1$. $S_1 = 1 \leq 2-1=1$;

Conditional (mean inductive):

$$S_k \leq 2 - \frac{1}{k} \Rightarrow S_{k+1} \leq 2 - \frac{1}{k+1}.$$

$$S_k \Rightarrow S_{k+1} \leq \frac{1}{(k+1)^2} + 2 - \frac{1}{k} = 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right)$$

$$\frac{1}{k} - \frac{1}{(k+1)^2} > \frac{1}{k} - \frac{1}{k(k+1)} = \frac{1}{k+1}$$

$$S_{k+1} \leq 2 - \frac{1}{k+1}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2. (30 points) Let us consider the arithmetic sequence with the recursive definition $a_{k+1} = a_k + d$.

1) Give the formula for a_n through a_1 and d and prove it by the method of mathematical induction.

2) By the same method to prove the formula for the sum of n terms:

$$S_n = a_1 + \cdots + a_n = \frac{(2a_1 + d(n-1))n}{2}.$$

1. $a_k = a_1 + d(n-1)$

$a_1 = a_1$

$a_k = d(k-1) + a_1 \Rightarrow a_{k+1} = a_1 + dk$;

$S_{k+1} = S_k + a_{k+1} = \cancel{(a_1 + dk)}$

$= (2a_1 + d(k-1))$

$a_{k+1} = a_k + d =$

$= a_1 + dk + d = a_1 + d(k$

2) $S_k = \frac{(2a_1 + d(k-1))k}{2} \Rightarrow S_{k+1} =$

$= \frac{(2a_1 + d(k-1))k}{2} + a_1 + dk$

$= a_1(k+1) + \frac{d(k+1)k}{2}$.

3. (30 points) Prove the tautology

$$((P \wedge Q) \vee (\sim P \wedge \sim Q)) \Leftrightarrow ((P \vee \sim Q) \wedge (\sim P \vee Q)).$$

by 2 methods:
-using True tables;
- using Propositional Laws without Tables.

$a \Leftrightarrow B$

1.

P	Q	a	B	$a \Leftrightarrow B$
T	T	T	T	T
F	F	T	T	T
F	T	F	F	T
T	F	F	F	T

2. $B \Leftrightarrow [(P \wedge \sim P) \vee (\sim Q \wedge \sim P)] \vee$

$[(P \wedge Q) \vee (\sim Q \wedge Q)$ distr. laws + commute.

$\Leftrightarrow [F \vee (\sim P \wedge \sim Q)] \vee [F \vee (P \wedge Q)]$

$P \wedge \sim P$
$Q \vee \sim Q$
$F \wedge Q$
$F \vee Q$

4. (30 points) 1.Transform to the convenient form the formula.

$$\sim [(P \Rightarrow \sim (Q \wedge \sim R)) \Leftrightarrow (\sim S \Rightarrow T)].$$

Simplify the answer.

2. Transform it a formula with only conjunctions and negations.

1. $(a \Leftrightarrow B) \Leftrightarrow (a \wedge B) \vee (\sim a \wedge \sim B)$

$\sim (a \Leftrightarrow B) \Leftrightarrow (\sim a \vee \sim B) \wedge (a \vee B)$

$(a \Rightarrow B) \Leftrightarrow \sim a \vee B$; $\sim (a \Rightarrow B) \Leftrightarrow a \wedge \sim B$

$\ldots [\sim (P \Rightarrow \sim (Q \wedge \sim R)) \vee (\sim S \Rightarrow T)] \wedge$

$[(P \Rightarrow \sim (Q \wedge \sim R)) \vee \sim (\sim S \Rightarrow T)] \Leftrightarrow$

$[(P \wedge (Q \wedge \sim R)) \vee (S \vee T)] \wedge$

$[(\sim P \vee (\sim Q \vee R)] \vee (\sim S \wedge \sim T)) \Leftrightarrow$

$[(P \wedge Q \wedge \sim R) \vee S \vee T] \wedge [(\sim P \vee \sim Q \vee R) \wedge \sim S \wedge$

2. $\sim [\sim [P \wedge Q \wedge \sim R) \wedge \sim S \wedge \sim T] \wedge [\sim (P \wedge Q \wedge \sim R) \wedge \sim S \wedge$

5. (30 points) Determine wether each of the following is Tautology, Contradiction, or neither. Don't use True Tables.

2) $((P \wedge \sim Q) \Rightarrow P) \Leftrightarrow (P \Rightarrow Q)$; neither

1) $((P \wedge \sim Q) \Rightarrow Q) \Leftrightarrow (P \Rightarrow Q)$; tautology

3) $\sim (\mathbf{F} \Rightarrow P)$.

1) $((P \wedge \sim Q) \Rightarrow Q) \Leftrightarrow \sim(P \wedge \sim Q) \vee Q \Rightarrow$

$\Leftrightarrow \sim P \vee Q \vee Q \Leftrightarrow \sim P \vee Q \Leftrightarrow P \Rightarrow Q.$

2) $\sim(P \wedge \sim Q) \vee P \Rightarrow \sim P \vee \sim Q \vee P \Leftrightarrow T$

neither $\quad T \not\Leftrightarrow P \Rightarrow Q$

3) $F \wedge P = F \quad$ contradiction

6. (30 points) 1. Prove the tautology

$$(\forall x)[P(x) \Rightarrow Q(x)] \Rightarrow [(\forall x)P(x) \Rightarrow (\forall y)Q(y)] \qquad (*)$$

2. Is it

$$(\forall x)[P(x) \Rightarrow Q(x)] \Leftarrow [(\forall x)P(x) \Rightarrow (\forall y)Q(y)]$$

tautology? Give detailed explanations.

1. $A \Rightarrow B$. We need show that if $A = T$ then $B = T$.

for some P, Q,

Let $A = T$: If $P(x_0) = T$ then $Q(x_0) = T$. (1)

What is mean $B = T$. If $P(x) \equiv T$ then $Q(x) \equiv T$ (2).

Proof. ~~If~~ Take any P, Q. ~~If~~

- If they don't satisfy (1) then $A = F$ and (*) is True.

- If ~~they~~ (1) ~~isn't true~~ is true, $A = T$ ~~and (2) isn't true ($P \equiv T$ but $Q \not\equiv T$) then B~~ and $P(x) \equiv T$ then $Q(x) \equiv T$.

- If $P(x) \not\equiv T$ then $B = T$.

2. Let $P(x) \not\equiv T, F$ and $Q(x) = \sim P(x)$, Then $B = T$ but $A = F$ (if $P(x_0) = T$ then $Q(x_0) = F$, $P(x_0) \Rightarrow Q(x_0) = F$)

7. (30 points) State using predicates and quantifiers and prove on this language that for natural m, n if $3|(mn+1)$ then $3|(m+n)$.

2. Is the inverse Theorem True?

$$3|(mn+1) \Rightarrow 3|(m+n).$$

Lemma. $(\forall n)\ 3|n \vee (\exists k)\ n = 3k+1 \vee (\exists l)\ n = 3l-1$

1. $3|n \vee 3|m \Rightarrow 3 \nmid mn+1.$

2. $n = 3k+1 \vee m = 3l+1 \Rightarrow mn = 3p+1 \Rightarrow 3 \nmid (mn+1)$

The same $n = 3k-1 \vee m = 3l-1.$

3. $n = 3k+1 \wedge m = 3l-1 \Rightarrow mn = 3p-1 \Rightarrow 3|mn.$

$\qquad$ —— " —— $m+n = 3r \Rightarrow 3|m+n$

4. The inverse is False: $3|m \wedge 3|n \Rightarrow$
$3|m+n \wedge 3 \nmid mn+1$

8. (30 points) Prove that $-x^2 - x + 6 < 0$ iff $x < -3 \vee x > 2$. Use predicates at Proof.

$-x^2 - x + 6 < 0 \Leftrightarrow x < -3 \vee x > 2.$

$-(x+3)(x-2) < 0 \Leftrightarrow [(x+3) > 0 \wedge (x-2) > 0]$

$[(x+3) < 0 \wedge (x-2) < 0] \Leftrightarrow (x-2) > 0 \wedge (x+3) <$

9. (30 points) 1. Give predicative definitions what does mean that a sequence is

- ~~restricted;~~ Bounded
- convergent.

2. Prove the any convergent sequence is bounded.

$(\exists A)(\exists B)(\forall n)\ A < x_n < B$ | or $(\exists M)(\forall n)\ |x_n| < M$

$(\exists L)$ $(\forall \varepsilon)\ \varepsilon > 0 \Rightarrow [(\exists N)(\forall n)\ n > N \Rightarrow |x_n - L| < \varepsilon]$

2. x_n - convergent. Then

~~$(\exists L)(\forall N)\ n > N \Rightarrow |x_n - L| < \varepsilon$~~

$(\exists L)$ ...

Take L and $\varepsilon = 1$. Then

$$L - 1 < x_n < L + 1 \text{ for } n > N(1).$$

Finite number of x_n, $n < N$, is bounded

$(\exists C)(\exists D)\ n < N \Rightarrow C < x_n < D.$

Take $A = \min(L-1, C)$, $B = \max(L+1, D)$.

10. (30 points) 1) Give the predicative definition that a sequence is
-unbounded;
-divergent.
2) Find limits if $n \to \infty$ of the following sequences, if they exist, and prove the existence or non existence, using directly the definitions:

$$\frac{n-1}{n(n+1)}; \frac{3^n - 2^n}{2^{2n} + 5^n}, \cos n.$$

1. $\lim\limits_{n\to\infty} \frac{n-1}{n(n+1)} = 0$

$|x_n - 0| = \frac{n-1}{n(n+1)} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon; \quad N(\varepsilon) = \left[\frac{1}{\varepsilon}\right]$

2. $\lim = 0$

$|x_n - 0| < \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n < \varepsilon, \quad n \ln\frac{3}{4} < \ln \varepsilon$

$n(\ln 3 - \ln 4) < \ln \varepsilon$

$n \ln 4 > n(\ln 4 - \ln 3) > \cancel{\ln} - \ln \varepsilon$

$N(\varepsilon) = \left[-\frac{\ln \varepsilon}{\ln 4}\right].$

3. $\{\cos n\}$, Subsequences $(-1)^n$, 1^n.

$\varepsilon = \frac{1}{2}$. $(\forall L)$ $L - \varepsilon < x < L + \varepsilon$

can't contain both ± 1.

$$\cancel{(\exists a} \; (\forall a)(\forall B)(\exists n) \; x_n \geq B \vee x_n \leq a.$$

$$(\forall L)(\exists \varepsilon) \; \varepsilon > 0 \wedge \left[(\forall N)(\exists n) \; n > N \wedge |x_n - L| \geq \varepsilon\right]$$