Lecture 9.

Division with a remainder.

Sect. 1.7

$a, b \in \mathbb{Z}$

$b = aq + r$, $0 \le r < |a|$.

q - quotien, r - remainder

Simple problems.

1. For $b \in \mathbb{N}$, $a = 6$, the remainder in the division on 6 is equal 5.
   What will be the remainder in the division a on 3?

$b = 6q + 5 \Rightarrow b = 3(2q+1) + 2$ ← remainder − 2

2. Which remainders are possible in the division $b = n^2$ on 5.

[Hint]. It's enough to consider $n = 0, 1, 2, 3, 4$

$n = 5q + r$, $n^2 = (5k + r)^2$.

Answer: 0, 1, 4.

$n = 5q + r$, $r = 0, 1, 2, 3, 4 \Rightarrow n^2 = 5\tilde{q} + r^2$.

Remainders for r^2 are 0, 1, 4.

3. Numbers ~~from~~ only "9".

$9\dots9$.

Theorem. 1) For each prime $p \neq 2,5$ there is such l that

$$p \mid \underbrace{9\dots\dots9}_{l \text{ times}}.$$

2) There is $l < p$.

Proof. Theorem $\Leftrightarrow$ for some l

$$p \mid 10^{l}-1.$$

For $10, 10^2, \dots, 10^{p}$ take the remainders for division of p:

$$r_1, r_2, \dots, r_{p}.$$

We have $0 < r_j < p$ since $p \nmid 10^{j}$.

Then between r_j there 2 equal ones since we have p natural numbers which can take only $p-1$ values (so many numbers $0<r<p$).

Let $r_i = r_j,\ i<j$. Then

$$p \mid 10^j - 10^i \Rightarrow p \mid 10^i(10^j-1)$$

$$\Rightarrow p \mid (10^{j-i}-1)$$

since $p \nmid 10$.

So $p \mid (10^l - 1)$, where $l = j-i$.

Other words $p \mid \underbrace{9\ldots9}_{l}$.

From the construction follows that $l < p$. $\blacksquare$

Decimal numbers:

$$\cancel{\overline{a_k \ldots a_2 a_1} = a_1 + a_2 10 + a_3 10^2 + \ldots + a_k}$$

$$\overline{a_k \ldots a_1 a_0} = a_0 + a_1 10 + \ldots + a_k 10^k.$$

Decimal fractions:

$$\overline{.b_1 \ldots b_k} = b_1 10^{-1} + \ldots + b_k 10^{-k}.$$

Predicats and quantifiers Sect. 1.3.

Informally: Predicat (or open sentence) is a Sentence in which the subject is not fixed but can be any element of a set (subject field or universe).

We consider predicats a function $P(x)$ which values are propositions and $\underset{\sim}{x}$ are elements of a subject field $\underset{\sim}{X}$.

So $P(x)$ ~~for~~ a fixed $x \in X$ can be either True or False.

Examples:

1. n is even. $(X = \mathbb{N})$

2. $1 < x \leq 2$ $(X = \mathbb{R})$

3. n is a square $(X = \mathbb{Z})$

4. ~~$1 \geq x$~~ $x \geq 0$ $(X = \mathbb{R})$

5. ~~(x,y) lies (a,b)~~ (x,y) lies on the line $2x - 3y + 5 = 0$. $(X = \mathbb{R}^2)$

6. x is a root of $x^2 - 3x + 2$. $(X = \mathbb{R})$

7. The ~~stage~~ a student has Major at Math (X - students of Rutgers)

8. This textbook is at our library (X - own library).

9. She is my dauther (X - my children).

etc.

There can be several subject variables with different subject fields:

$P(x_1, \dots, x_n)$.

Examples:

1. $a \mid b$ $(\mathbb{N} \times \mathbb{N})$
2. $x < y$ $(\mathbb{R} \times \mathbb{R})$
3. Line l passing through point a.
   $\{\text{lines}\} \times \{\text{points}\}$
4. This university is at New Jersey.

2 operations at Algebra of Predicats.

Universal quantifier $(\forall x)P(x)$ - {for all x $P(x)$}

Existential quantifier $(\exists x)P(x)$ - {exists such x that P}.

Application of quantifiers decreseas the number of subject variables on 1.

Examples.

$P(n)$ = n is even

$$P(n) \Leftrightarrow \underline{(\exists k)} \quad n = 2k \qquad \mathbb{N}\times\mathbb{N}$$

$$a|b \Leftrightarrow (\exists q) \quad b = aq \qquad (\mathbb{N}\times\mathbb{N}\times\mathbb{N})$$

Division ~~a~~ with a remainder - (Theorem)

$$(\forall a)(\forall b)(\exists q)(\exists r) \quad b = aq + r \wedge 0 \le r < |a|)$$

$$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$$

n is a squere $\Leftrightarrow \exists k \quad n = k^2$, ~~$\mathbb{Z}\times\mathbb{Z}$~~ $\mathbb{N}\times\mathbb{N}$