Construction.

Let $\sigma = (\sigma_1, \dots, \sigma_n)$ is a combination of T and F („word"). σ define lines at True table. Let us agree that

$$P^T = P$$
$$P^F = \sim P,$$

and $C_\sigma(P_1, \dots, P_n) = P_1^{\sigma_1} \wedge P_2^{\sigma_2} \wedge \dots \wedge P_n^{\sigma_n}$ is a propositional form.

Ex. $\sigma = (T, F, F, T)$.

$C_\sigma(P_1, P_2, P_3, P_4) = P_1 \wedge \sim P_2 \wedge \sim P_3 \wedge P_4$.

Lemma. $C_\sigma(P_1, \dots, P_n)$ is Truth iff $\sigma = (\sigma_1, \dots, \sigma_n)$ is the True value of propositions $(P_1, \dots, P_n)$.

Theorem. Let ~~f~~ $f(P_1, \dots, P_n)$ is a proposional function and

$$C_f(P_1, \dots, P_n)$$

is the disjunction of $C_\sigma(P_1, \dots, P_n)$ for such σ that if σ is true value of $P_1, \dots, P_n$ then F is Truth.

Then

$$C_f(P_1, \dots, P_n)$$

is equivalent to $f(P_1, \dots, P_n)$.

Proof. Let $\sigma = \overbrace{(\sigma_1, \dots, \sigma_n)}^{\text{are such true values of } P_1 .. P_2 \text{ th}}$ in $f(\sigma)$ is Truth then one of terms (and only one!) $C_\sigma(P_1, \dots, P_n)$ is Truth and the whole disjunction will be Truth.

If $f(\sigma) = F$ then all terms and whol disjunction is F. ◻

Examples.

1. Find a ~~f~~ propositional form for the question at the problem about Alice.

Solution. $f(P_1, P_2)$ is Truth for

$\sigma = (T,T), (F,F)$.

So $$C_f(P_1, P_2) = (P_1 \wedge P_2) \vee (\sim P_1 \wedge \sim P_2).$$

2. $f(P_1, P_2, P_3)$ is T if ~~at~~ the most of P_j is T. Find an equiv. formula.

$$C_F(P_1, P_2, P_3) = (P_1 \wedge P_2 \wedge P_3) \vee (\sim P_1 \wedge P_2 \wedge P_3)$$
$$\vee (P_1 \wedge \sim P_2 \wedge P_3) \vee (P_1 \wedge P_2 \wedge \sim P_3).$$

Definition. A system of propositional operations is called complet if each connective can be ~~presen~~ presented an equivalent formula through there operations.

{conjunction, disjunction, negation} is a complet system

We have an example $\{\wedge, \vee, \sim\}$ of complet system of operations.

There are other complet systems.

Proposition. Any of pairs $\{\wedge, \sim\}$ or $\{\vee, \sim\}$ is an complet.

Proof. To prove that the pair $\{$conjuction, negation$\}$ is complet we need to present disjunction $\vee$ by an equivalent formula from only $\wedge, \sim$.

From de Morgan Law $\sim(P\vee Q)\equiv \sim P\wedge\sim$ we have

$$\boxed{P\vee Q \equiv \sim(\sim P\wedge \sim Q)}$$

So we express disjunction through ~~disj~~ conjuction and negation.

Ex. $\sim(P\vee(Q\wedge R))\equiv \sim P\wedge \cancel{\sim(\sim P\wedge \sim(Q\wedge R}$

$\sim P\wedge\sim(Q\wedge R)$

No $\vee$ in the final formula!

Similarily:

$$\boxed{P \wedge Q \equiv \sim(\sim P \vee \sim Q)}$$

and we can represent any propositional function only through negations and disjunctions.

In our example:

$$\sim(P \vee (Q \wedge R)) \equiv \sim(P \vee \sim(\sim Q \vee \sim R))$$

No conjunctions!

Let us define a new operation:

Scheffer's operation

$$\boxed{P \square Q \overset{def}{\equiv} \sim P \wedge \sim Q}$$

Proposition. Scheffer's operation alone is a complete system.

Proof. It sufficient to represent through ~~th~~ this operation negation and conjunction (or disjunction!). We have

$$\sim P \equiv P \square P;$$

$$P \wedge Q \equiv \sim P \square \sim Q \equiv (P \square P) \square (Q \square Q)$$

Boolean algebra.

Arithmetic on the set of 2 elements: $\{0,1\}$. It's arithmetic modulus 2: replace numbers by remainders of the division of 2:

$0+0=0$ $\qquad$ $0\cdot 0=0\cdot 1=1\cdot 0=0$

$0+1=1+0=1$ $\qquad$ $1\cdot 1=1$.

$1+1=0$

Functions $f(x_1,\dots,x_n)$ on $\{0,1\}$ are Boolean functions.

There is a natural correspondence between Boolean and Propositional functions:

$$T \leftrightarrow 1 \qquad \sim P \leftrightarrow x+1$$

$$F \leftrightarrow 0$$

$$P\wedge Q \leftrightarrow xy$$

Which arithmetical operation corresponds to the disjunction?

$$P\vee Q \equiv \sim(\sim P \wedge \sim Q) \leftrightarrow ((x+1)(y+1)+1)$$

$P \leftrightarrow x$

$Q \leftrightarrow y$

$$= \cancel{xy}\ xy+x+y.$$

~~Ch~~ So connectives corresponds to polynomials mod 2.

Conditional connective (Sect. 1.2)

(or implication)

$P \Rightarrow Q$

corresponds many ~~sign~~ grammatical constructions:

"if P then Q"

"from P follows Q"

"Q is necessary condition of P".

Ex. P - "a number is multiple of 6"

Q - —— " —— 3".

Again we are interesting just True values but don't consider any causal connections between P and Q. and it ~~a~~ often looks ambigouss.

We define $P \Rightarrow Q$ by its True Table:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The table follows to the principle « From Truth follows ~~of~~ only Truth but from Lie can follow both Truth and Lie. »

Properties:

1. $P \Rightarrow Q \equiv \sim P \vee Q$
2. $\sim (P \Rightarrow Q) \equiv P \wedge \sim Q$
3. $P \Rightarrow (Q \Rightarrow R) \equiv (P \wedge Q) \Rightarrow R$
4. $P \Rightarrow (Q \wedge R) \equiv (P \Rightarrow Q) \wedge (P \wedge R)$
5. $(P \vee Q) \Rightarrow R \equiv (P \Rightarrow R) \wedge (Q \Rightarrow R)$.