Lecture 27.

Ha 12.

1. a) $\lim\limits_{n \to \infty} \frac{(-1)^n}{n^2} =$

1) Guess $\lim = 0$

2) Analysis $|x_n - L| = \frac{1}{n^2} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n^2$.

$N(\varepsilon) = \left[\sqrt{\frac{1}{\varepsilon}}\right]$.

Variants: $\Leftrightarrow x_n = \frac{1}{n^2}$; $\frac{1}{n^2} < \frac{1}{n} \Rightarrow N(\varepsilon) = \left[\frac{1}{\varepsilon}\right]$;

$\frac{1}{n^2}$ - a subsequence of $\frac{1}{n}$. $N(\varepsilon)$ isn't unique!

b) $\lim\limits_{n \to \infty} \frac{1}{3n-2} = 0$

$\left|0 - \frac{1}{3n-2}\right| = \frac{1}{3n-2} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < 3n-2 \Leftrightarrow \frac{\frac{1}{\varepsilon}+2}{3} < n$

$N(\varepsilon) = \left[\frac{\frac{1}{\varepsilon}+2}{3}\right]$

Variant: $\frac{1}{3n-2} < \frac{1}{4n} \Rightarrow N(\varepsilon) = \left[\frac{1}{4n}\right]$.

c) $\lim\limits_{n \to \infty} \frac{\cos n}{n} = 0$

$|x_n - L| = \frac{|\cos n|}{n} < \frac{1}{n} \Rightarrow N(\varepsilon) = \left[\frac{1}{\varepsilon}\right]$.

d) $\lim_{n\to\infty} \frac{n^2}{n^2+3} = 1$

$|L - x_n| = \frac{3}{n^2+3} < \frac{3}{n^2} < \varepsilon < \frac{3}{n} \Rightarrow N(\varepsilon) = \left[\frac{3}{\varepsilon}\right].$

e) $\frac{2^n}{3}$ - disverges

$(\forall L) (\exists \varepsilon)\ \varepsilon > 0 \wedge \{(\forall N)(\exists n)\ n > N \wedge |x_n - L| \geq \varepsilon$

See Lemme below: $x_n = \frac{2^n}{3}$ is unbounded from above.

$\frac{2^n}{3} \geq M \Leftrightarrow n \geq \log_2 3M.$ So we can take $n(M) = [\log_2 3M] + 1.$

f) $n!$ - disverges

Lemma. $n!$ is unbounded from above!

$(\forall M)(\exists n)\ n! \geq M.$

Take $n = [M] + 1.$ Then $n! > n > M.$

Corollary. $n!$ disverges.

$n(M)$

~~Proof~~. Take arbitrary L, N and $\varepsilon = 1$. Then take $M > \max(L+1)$.

Then for any $n > N$ and $n > n(M)$ at Lemma we have

$$|x_n - L| > \varepsilon = 1 \wedge n > N.$$

[Using Theorem we could be deduce e), f) from Lemma.

2. $$\lim_{n\to\infty} \frac{3n^2-3n+1}{6n^2+2n+1} = \lim_{n\to\infty} \frac{3-\frac{3}{n}+\frac{1}{n^2}}{6+\frac{2}{n}+\frac{1}{n^2}} = \frac{1}{2}.$$

$$\lim_{n\to\infty} \frac{2n^2-5}{3n^3+6} = \lim_{n\to\infty} \frac{\frac{2}{n}-\frac{5}{n^3}}{3+\frac{6}{n^3}} = \frac{0}{3} = 0.$$

$$\lim_{n\to\infty} \frac{2^n-3^n}{4^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n - \left(\frac{3}{4}\right)^n = 0.$$

$\lim_{n\to\infty} q^n = 0$ $\qquad |q^n| < \varepsilon \Leftrightarrow -\log_{(1/q)} n <$

$|q| < 1$ $\qquad q>0$ $\qquad \varepsilon \Leftrightarrow n$

$0<q<1$ $\quad <0$ $\qquad < -\log_{1/q}$

$q^n < \varepsilon \Leftrightarrow n \ln q < \ln \varepsilon$

$\Leftrightarrow n > \frac{\ln \varepsilon}{\ln q}.$ $\qquad N = \left[\frac{\ln \varepsilon}{\ln q}\right].$

$x_n = n^3$ - diverges, since n^3 - unbonded from above (see above).

$x_n = \cos n\pi = (-1)^n$ diverges.

3. Theorem. The limit of a convergent sequence is unique.

$$\lim_{n\to\infty} x_n = L \wedge \lim x_n = M \Rightarrow L = M.$$

Proof by Contradiction. Let

$$(\forall\varepsilon)\ \varepsilon>0 \Rightarrow \{(\exists N_1)(\forall n)(n>N_1 \Rightarrow |x_n-L|<\varepsilon) \wedge$$

$$(\exists N_2)$$

$$\wedge\ (n>N_2 \Rightarrow |x_n-M|<\varepsilon)\} \wedge L\neq M \Leftrightarrow$$

$$(\forall\varepsilon)\ \varepsilon>0 \Rightarrow \{(\exists N)(\forall n)\ n>N \Rightarrow (|x_n-L|<\varepsilon \wedge$$

$$\wedge\ |x_n-M|<\varepsilon \wedge L\neq M\} \Rightarrow |L-M|<2\varepsilon$$

$$\left[\,|L-M| = |(L-x_n)+(x_n-M)| < 2\varepsilon\,\right]$$

$$(\forall\varepsilon)\ \varepsilon>0 \Rightarrow |L-M|<2\varepsilon \Rightarrow L-M=0.$$ Contradiction.