2. $x_n = \frac{1}{n^2}$.

1) $\lim = 0$.

2) $|x_n| = \frac{1}{n^2} \leq \frac{1}{n}$. So again we can take $N(\varepsilon) = \left[\frac{1}{\varepsilon}\right]$

3) $n > N(\varepsilon) = \left[\frac{1}{\varepsilon}\right] \Rightarrow |0 - x_n| = \frac{1}{n^2} < \frac{1}{n}$

$< \frac{1}{\left[\frac{1}{\varepsilon}\right]} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$.

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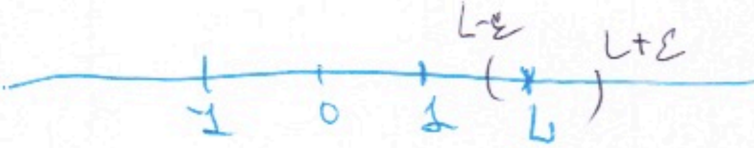
3. $x_n = (-1)^n$: $-1, 1, -1, \ldots$

1) Guess: No limit! (disverges)



$(\forall L)(\exists \varepsilon)\ \varepsilon > 0 \wedge \{(\forall N)(\exists n)[n > N \wedge |L - x_n| \geq \varepsilon]\}$

$\boxed{\varepsilon(L)}$ Take such $\varepsilon = \varepsilon(L) > 0$ s.t.

$\boxed{|L - 1| > \varepsilon \vee |L + 1| > \varepsilon}$ (*)

Then a "half" of x_n "will" $|x_n - L| > \varepsilon$

Let $|L - 1| > \varepsilon$.

So 1 is outside of ε-neighborhood of L.
The case of $|L+1| > \varepsilon$ is considered similarly

Remark. We can ~~a~~ take ~~$\varepsilon =$~~ $\varepsilon(L) \equiv \frac{1}{2}$ (independent of L!). Then $x = -1, 1$ can't be both ~~lie~~ simultaneously ~~in~~ / ~~at~~ $|L - x| < \varepsilon$.

Why?

If ~~$|L+1|>$~~ $|L-1] > \varepsilon$ for any ($\forall N$)

we cane take any ~~odd~~ even n

and then $x_n = 1 \wedge |L - x_n| > \varepsilon$.

$$\lim_{n \to \infty} \frac{1}{2^n} = L \ ?$$

1) $L = 0$

2) $|L - x_n| = \frac{1}{2^n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < 2^n$

$$N(\varepsilon) = [\log_2 \frac{1}{\varepsilon}] = [-\log_2 \varepsilon].$$

Some theoretical facts.

1. Theorem. Let

$$\lim_{n\to\infty} x_n = L \wedge \lim_{n\to\infty} y_n = M$$

and $z_n = x_n + y_n$. Then

$$\lim_{n\to\infty} z_n = L + M.$$

Proof. We need to show that

$$(\forall\varepsilon)\ \varepsilon>0 \Rightarrow \{(\exists N)(\forall n)(n>N \Rightarrow |L+M-z_n|<\varepsilon)\}$$

Take $\varepsilon_1 = \frac{\varepsilon}{2}$.

Since $\lim_{n\to\infty} x_n = L$, there is $N_1(\varepsilon_1)$ s.t.

$$(\forall n)\ (n > N_1(\varepsilon_1) \Rightarrow |L - x_n| < \varepsilon_1 = \frac{\varepsilon}{2}.$$

and there is $N_2(\varepsilon_1)$ s.t.

$$(\forall n)\ (n > N_2(\varepsilon_1) \Rightarrow |M - y_n| < \varepsilon_1 = \frac{\varepsilon}{2}.$$

Take $N(\varepsilon) = \max\left(N_1\left(\frac{\varepsilon}{2}\right), N_2\left(\frac{\varepsilon}{2}\right)\right)$.

Then $(\forall n)\ n > N(\varepsilon) \Rightarrow |L - x_n| < \frac{\varepsilon}{2} \wedge |M - y_n| < \frac{\varepsilon}{2} \Rightarrow$

$$\Rightarrow |L+M-x_n-y_n| \le |L-x_n|+|M-y_n| < \varepsilon. \blacksquare$$

Other theorems:

$$\lim_{n\to\infty} x_n y_n = L\cdot M;$$

$$\lim_{n\to\infty} c\, x_n = cL;$$

$$M \neq 0 \Rightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{L}{M}.$$

Ex. $$\lim_{n\to\infty} 3-\frac{2}{n}+\frac{3}{n^3} = 3.$$

$$\cancel{\lim_{n\to\infty} 2^n}$$

1) $$\lim_{n\to\infty} \frac{2n^3-3n^2+n}{5n^3-1} =$$

$$\frac{2}{5}.$$

$$= \lim_{n\to\infty} \frac{2-\frac{3}{n}+1}{5-\frac{1}{n^3}} = \frac{2}{5}$$

2) $$\lim \frac{2^n-5^n}{13^n \cancel{+1}} = 0$$

$$= \left(\frac{2}{13}\right)^n - \left(\frac{5}{13}\right)^n.$$

<u>Theorem.</u> Monotone bounded sequence has a limit. (without proof).

$\{x_n\}$ (increasing), bounded from above,

$$\{(\forall n_1)(\forall n_2)\ n_1 < n_2 \Rightarrow x_{n_1} \le x_{n_2}\} \wedge$$

$$\{(\exists M)(\forall n)\ x_n \le M\} \Rightarrow$$

$$(\exists L)\ \lim_{n\to\infty} x_n = L.$$

Idea: Take $y_1 = x_1$, $y_2 = L$, ..., $y_{k+1} =$,

~~Take y_{k+1} s.t.~~ 

Continue divide segments on halfs and choose unique half with <u>infinite</u> number of points x_n.

These segments have unique joint point L.

$$L = \lim_{n\to\infty} x_n.$$

Theorem. Each convergent sequence is bounded.

Proof. $\lim_{n\to\infty} x_n = L$. Take any $\varepsilon>0$.

Let $\varepsilon=1$. $\underset{N(1)}{(\exists N)}(\forall n)\ n>N \Rightarrow |x_n-L|<1$.

There $(\forall n)\ n>N \Rightarrow \underline{L-1<x_n<L+1}$

So subsequence $\{x_n, n\geq N\}$ is bounded. (By $L-1$ from below and $L+1$ from above).

We can replace $(L-1, L+1)$ on (a, B) s.t. that $a<x_n<B$.

Definition. $\{y_m\}$ is a subsequence of $\{x_n\}$ iff there is monotone $f(m)=n$ (s.t. $m_1<m_2 \Rightarrow f(m_1)<f(m_2)$) s.t.

$(\forall m)\ y_m = x_{f(m)}$.

Theorem. If $\lim_{n\to\infty} x_n = L$ then all
subsequences have the same limit.

Hint. $(\forall \varepsilon)$. Find $N(\varepsilon) : (\forall n)\, n > N(\varepsilon) \Rightarrow |x_n - L| < \varepsilon$.
For $\{y_m\}$ take $M(\varepsilon)$ s.t. $m > M(\varepsilon) \Rightarrow f(m) > N(\varepsilon)$

Theorem. If $\{x_n\}$ has 2 subsequences $\{y_m\}, \{z_p\}$ with different limits R and S, then $\{x_n\}$ has no limit.

Hint. Take $\varepsilon < \frac{|R-S|}{2}$.

Theorem. Let $\lim_{n\to\infty} x_n = L \wedge \lim_{n\to\infty} y_n = L \wedge$
$(\forall n)\ x_n \leq z_n \leq y_n$ then

$$\lim_{n\to\infty} z_n = L.$$

[Theorem on 2 policmen].