Lecture N2,

6h) $S_n = \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

(i) $n=1$. $\frac{1}{2} = \frac{1}{2}$

(ii) $S_k = 1 - \frac{1}{(k+1)!} \overset{?}{\Rightarrow} S_{k+1} = 1 - \frac{1}{(k+2)!}$

$P \Rightarrow Q$

$P \Rightarrow S_{k+1} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} =$

$= 1 - \left(\frac{(k+2) - (k+1)}{(k+2)!} \right) = 1 - \frac{1}{(k+2)!}$

6i) $P_n = \prod_{i=1}^{n} \left(1 - \frac{1}{i+1}\right) = \frac{1}{n+1}$

(i) $n=1$, $\frac{1}{2} = \frac{1}{2}$.

(ii) $P_k = \frac{1}{k+1} \Rightarrow P_{k+1} = \frac{1}{k+1}\left(1 - \frac{1}{k+2}\right) =$

$= \frac{1}{k+1} \cdot \frac{k+1}{k+2} = \frac{1}{k+2}$, 7(f) $9 \mid 10^{n+1} + 3 \cdot 4^{n-1} + 5$

(i) $n=1$ $\quad 9 \mid (100+3+5) \Leftrightarrow 9 \mid 108$ True.

(ii) $9 \mid \underbrace{10^{k+1} + 3 \cdot 4^{k-1} + 5}_{a_k} \Rightarrow$

$$a_{k+1} = 9 \cdot 10^{k+1} + 10^{k+1} + 9 \cdot 4^{k-1} + 3 \cdot 4^{k-1} + 5$$

$$= a_k + 9q \Rightarrow 9 \mid a_{k+1}$$

7(h) $3^n \geq 1 + 2^n$

(i) $n=1$ $\quad 3 \geq 3$;

(ii) $3^k \geq 1 + 2^k \Rightarrow 3^{k+1} \geq 3 + 3 \cdot 2^k >$

$1 + 2^{k+1}$

7k)

$$S_n = \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

(i) $n=1;\ 1 \leq 1$

(ii) $S_k \leq 2 - \frac{1}{k} \Rightarrow S_{k+1} = S_k + \frac{1}{(k+1)^2} \leq$

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} \leq$$

$$\leq 2 - \left(\frac{1}{k} - \frac{1}{k(k+1)}\right) = 2 - \frac{1}{k+1}.$$

8e) $P_n = \prod_{2}^{n} \frac{i^2-1}{i^2} = \frac{n+1}{2n}$, $n \geq 2$

(i) $n=2 \quad \frac{3}{4} = \frac{3}{4}$

(ii) $P_k = \frac{k+1}{2k} \Rightarrow P_{k+1} = \frac{k+1}{2k} \cdot \frac{(k+1)^2-1}{(k+1)^2} =$

$$= \frac{k+2}{2(k+1)}$$

8f) $P_n = \prod_{i=1}^{n} \frac{1}{i} \leq 2^{-n}, \; n \geq 4$

(i) $n = 4$; $\frac{1}{24} \leq \frac{1}{16}$

(ii) $P_k \leq 2^{-k} \Rightarrow P_k \leq 2^{-k} \frac{1}{k+1} \leq 2^{-(k+1)}$
$k \geq 1$.

8c) $(n+1)! \geq 2^{n+3}, \; n \geq 5$

(i) $n = 5$ $\quad 6! = 720 \geq 2^8 = 256$ True

(ii) $\cancel{P}$ $(k+1)! \geq 2^{k+3} \Rightarrow (k+2)! \geq (k+2)2^{k+3} \geq$

$\geq \cancel{(k+2)} 2^{k+4}$.

2. Sequance $a_1, \ldots, a_n, \ldots$

1) monotone increasing if. $\mathbb{N}$

$(\forall i)(\forall j)\ i<j \Rightarrow a_i < a_j$.

2) isn't — " —

$(\exists i)(\exists j)\ i<j \wedge a_i \geq a_j$;

3) is not monotone, if

$(\exists i)(\exists j)(\exists k)(\exists l)\ i<j \wedge k<l \wedge a_i \geq a_j \wedge$
$\wedge\ a_k \leq a_l$.

4) bounded

$(\exists M)(\forall n)\ |a_n| < M$.

5) unbounded

$(\forall M)(\exists n)\ |a_n| \geq M$.

6) Bounded from above

$$(\exists M)(\forall n)\ a_n < M$$

7) Unbounded from above

$$(\forall M)(\exists n)\ a_n \geq M.$$

8) Unbounded from above but bounded from below

$$(\forall M)(\exists n)(\exists P)(\forall m)\ a_n \geq M \wedge a_m < P.$$