MA10 7ℓ) $(\forall x)\, x > 0 \Rightarrow$

$$(1+x)^n \geq 1 + nx$$

(i) $n=1$: $1+x \geq 1+x$ ✓

(ii) $(1+x)^k \geq 1+kx \Rightarrow$

$$(1+x)^{k+1} \geq (1+kx)(1+x) = 1+(k+1)x+kx^2 >$$

$$> 1+(k+1)x.$$

Extra problems.

1. $(\forall n)\, n > 11 \Rightarrow (\exists a)(\exists b)\ n = 4a+5b.$

$\mathbb{Z}_+$

(i) $n = 12$, $12 = 4\cdot 3$. $a=3, b=0$.

(ii) $k = 4a+5b \Rightarrow k+1 = 4\tilde{a}+5\tilde{b}$.

Crucial: $5-4=1$

1) $a>0$. Then $(a-1, b+1)$ works

2) $a=0 \Rightarrow b \geq 3$. $(\tilde{a}, \tilde{b}) = (a+4, b-3)$.

~~5·4-~~

$4\cdot 4 - 5\cdot 3 = 1$

2. For $k=1$ sets (1) and (2) have no common members.

The Fibbonacci numbers.

Fibbonacci sequence:

$f_1 = 1, f_2 = 1$

$f_n = f_{n-1} + f_{n-2}, \; n \geq 3 \quad (*)$

[Fibbonaci, 1202

Pair of rabbits; breed ~~x~~ if they are ~~older~~ breed if ~~2a~~ they are 2 months old or older and produce 1 pair / month.

$f_3 = 2$

$f_4 = 3$

$f_5 = 5$

$f_6 = 8$

spirals in plants with a pattern (phyllotaxis)

Proposition. $\sum_{i=1}^{n} f_i = f_{i+2} - 1. \quad (**)$

Proof. (i) $n=1$. $1 = 2-1$. True

(ii) $\sum_{i=1}^{k} \ldots = f_{k+2} - 1 \Rightarrow \sum_{i=1}^{k+1} \ldots = f_{k+2} - 1 + f_{k+1}$

$= (f_{k+1} + f_{k+2}) - 1 = f_{k+3} - 1.$

Let $\alpha = (1+\sqrt{5})/2$, $\beta = \frac{1-\sqrt{5}}{2}$

Roots of $x^2 - x - 1 = 0$

Golden section $f_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n)$

Proposition. $f_n > \alpha^{n-2}$, $n \geq 3$

~~(i) $n=1$~~,

Sequences of real numbers

$a_1, a_2, \ldots a_n, \ldots$

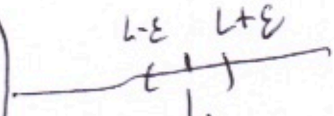
Ex. $\frac{1}{n}$; $\frac{(-1)^n}{n}$; $\frac{\sin n}{n}$; $3, 3, \ldots$, 2^n,

Principal definition:

$\mathbb{R}$ $\mathbb{N}$

$L = \lim_{n \to \infty} x_n$ iff

$(\forall \varepsilon)\ \varepsilon > 0 \Rightarrow \left((\exists N)\ (\forall n)\ n > N \Rightarrow |L - x_n| < \varepsilon \right)$



Negation

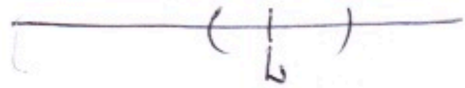
$L \neq \lim_{n \to \infty} x_n$ iff

$(\exists \varepsilon)\ \varepsilon > 0 \wedge \left((\forall N)\ (\exists n)\ n > N \wedge |L - x_n| \geq \varepsilon \right)$

The sequence disverges iff

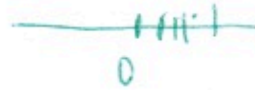
$(\forall L)$

Remark: $\quad |x_n - L| < \varepsilon \Leftrightarrow L-\varepsilon < x_n < L+\varepsilon$



Examples.

1. $x_n = \frac{1}{n}$. Guess: $L=0$



~~Prove~~ Proof: Take $(\forall\varepsilon)\ \varepsilon>0$

Analysis: $|x_n - 0| = \cancel{}\left|\frac{1}{n}-0\right| = \frac{1}{n}$

when $\frac{1}{n}<\varepsilon$? If. $\frac{1}{\varepsilon}<n$

So take $N(\varepsilon) = [\frac{1}{\varepsilon}]$
integer part.

Then $\frac{1}{\varepsilon}<n$ if $n>N(\varepsilon)$.

2. $x_n = \frac{1}{3n}$. Again $L=0$.

$\varepsilon>0$. ~~Whe~~ Which $N(\varepsilon)$ can we take?

$\frac{1}{3n}<\varepsilon \Leftrightarrow \frac{1}{3\varepsilon}<\varepsilon$. So $N(\varepsilon) = \left[\frac{1}{3\cdot\varepsilon}\right]$

We can always take $N(\varepsilon)$ greater –

3. $x_n = 3$. $L = 3$

$N(\varepsilon) = 1$

$|x_n - L| = 0$

4. $x_n = \frac{1}{2^n}$.

$L = 0$ $|x_n - L| = \frac{1}{2^n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < 2^n$

$N(\varepsilon) = [\log_2 \frac{1}{\varepsilon}] \Rightarrow n > N(\varepsilon) \Rightarrow 2^n < \frac{1}{\varepsilon}$.

5. $x_n = \frac{n}{n+1}$.

Guess. $L = 1$

$|L - x_n| = \frac{1}{n+1}$;

So we can take $N(\varepsilon) = [\frac{1}{\varepsilon}] + 1$ as for

$x_n = \frac{1}{n+1}$

Remark. $\frac{n}{n+1} = 1 - \frac{1}{n+1}$

6. $x_n = \frac{1}{n^2}$. $\lim_{n \to \infty} \frac{1}{n^2} = 0$

Reason: $0 < \frac{1}{n^2} < \frac{1}{n}$. Why?