Lecture 19.

Selective Problems from HW9.

1. Tautology

$$(\exists x)(P(x) \vee Q(x)) \Leftrightarrow ((\exists y) P(y) \vee (\exists z) Q(z)).$$

$$A \Leftrightarrow B.$$

We need to prove that either A, B are both True or they are both False.

If we prove it about True then automatically it's true about False:

A is True iff for some x_0 $P(x_0) \vee Q(x_0)$ is True. So $P(x_0) = T$ or $Q(x_0) = T$.

If $P(x_0) = T$ then $(\exists y) P(y) = T$.

-"- $Q(x_0) = T$ then $(\exists z) Q(z) = T$.

So $A \Rightarrow B$.

We can move in opposite direction:

Ex. If $Q(z_0) = T$ then $P(z_0) \vee Q(z_0) = T$ and

$(\exists x)(P(x) \vee Q(x)) = T$.

2. $(\forall x)(P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y) P(y) \Rightarrow (\forall z) Q(z))$

$A \Rightarrow B$

a) A - True iff for each x_0 if $P(x_0) = T$ then $Q(x_0) = T$

Then if $P(y) \equiv T$ then $Q(z) \equiv T$ and $A \Rightarrow B$.

b) $B \not\Rightarrow A$. If $P(y) \not\equiv T$ then $B \equiv T$ always. but A can be False (if $P(x) = T, Q(x) = F$).

Example. $P(x)$ - x is odd

$Q(x)$ - x is even.

B - T but A - F

2. $2^a 3^b 5^c \mid N$. N - product of 10 consequent numbers. Between at least

5 -	multiple	2
2 -	— " —	4
1 -	— " —	8
3 -	— " —	3
1	— " —	9
2	— " —	5

So $a \geq 8$

$b \geq 4$

$c \geq 2$.

3. 2 axioms:

a_1: $(\forall x_1)(\forall x_2)(\forall x_3)(\forall \ell)(x_1 \in \ell \wedge x_2 \in \ell \wedge x_3 \in \ell \wedge$
$x_1 \ne x_3 \wedge x_1 \ne x_2 \wedge x_2 \ne x_3) \Rightarrow x_1 < x_2 < x_3$.

a_2: $(\forall x)(\forall y)(\exists z)\ x < z < y$.

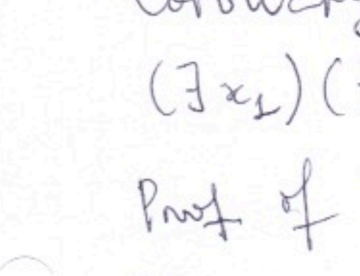
Lemma. For any finite set $x_1, \dots, x_n \in \ell$
$(\exists x_1)(\exists x_2)(\forall x_j)\ j \ne 1, j \ne 2 \Rightarrow x_j \in (x_1, x_2)$.

Corollary. For any finite set $\{x_j\}$ on ℓ
$(\exists x_1)(\exists x_2)\ \forall j\ j \ne 1,2 \Rightarrow x_j \in (x_1, x_2)$.

Proof of Theorem. ~~$\exists\{x_j\}$~~ $\{x \in \ell\}$ is infinite.

Contraposition. $\{x_j\}$ is finite. Take (x_1, x_2) as at Corollary. Then take $y \notin (x_1, x_2)$ and y isn't at our set.

4. Take $\ell \parallel m$ (exists on axiom of Parallel).



Let n intersects ℓ, m and ~~x, y ar~~ ~~$x \in \ell, y$~~ $x \in m, y \in \ell$ are intersection points.

Let $z \in m \wedge z = x$. Let $p \parallel n \wedge z \in p$.

Then $p \nparallel \ell$ (axiom parallel). Let $u = p \cap \ell$.

Then $xyuz$ – parallelogram.

5. [$\mathbb{R}$]

$(\forall x)(\forall y)\ xy<0 \Leftrightarrow (x<0 \wedge y>0) \vee (x>0 \wedge y<0).$

$a \Leftrightarrow B$

$B \Rightarrow a.$ $\quad B = B_1 \vee B_2.$ $\quad (B_1 \Rightarrow a \wedge B_2 \Rightarrow a) \Rightarrow B \Rightarrow a$

$a \Rightarrow B.$ By contradiction:

$\sim B \Rightarrow x=0 \vee y=0 \vee (x>0 \wedge y>0) \vee (x<0 \wedge y<0)$

$= C_1 \vee C_2 \vee C_3 \vee C_4$

$C_1 \vee C_2 \Rightarrow xy=0 \Rightarrow \sim a$

$C_3 \vee C_4 \Rightarrow xy>0 \Rightarrow \sim a$

$\sim B \Rightarrow \sim a.$ Contradiction.

6. $-x^2-2x+15>0 \Leftrightarrow -5<x<3$

$\Updownarrow$

$-(x+5)(x-3)>0 \Leftrightarrow -5<x<3 \Leftrightarrow -5<x \wedge x<3$

$a \Leftrightarrow B$

$B \Rightarrow (x+5)>0 \wedge (x-3)<0 \Rightarrow a$

$a \Rightarrow B.$ By contradiction

$\sim B \Rightarrow x<-5 \vee x>3 \Leftrightarrow x+5<0 \vee x-3>0 \Rightarrow$

$-(x+5)(x-3)<0 \Rightarrow \tilde{a}$ Contradiction

Mathematical Induction.

2 modifications.

1) If

 (i) $n_0 \in S$

 (ii) $(k \in S \wedge k \geq n_0) \Rightarrow (k+1) \in S$.

Then $(\forall n)\ n \geq n_0 \Rightarrow n \in S$.

2) If

 (i) $1 \in S$

 (ii) $\{1, 2, \ldots, k\} \in S \Rightarrow (k+1) \in S$.

Examples.

Prove.

1. $n < 2^n$ (i) $k = 1$ True

 (ii) Let $k < 2^k$. Then

 $2k < 2^{k+1}$. Since $2k \geq k+1$

 We have $(k+1) < 2^{k+1}$

2. $2^n > n^2$ for $n > 4$. Restriction is essential,

 (i) ($k = 4$ $16 \geq 16$)

 $n = 5$ $32 > 25$

(ii) Let $2^k > k^2$. Then

$$2^{k+1} > 2k^2$$

$$\left(\frac{k+1}{k}\right)^2 = \left(1+\frac{1}{k}\right)^2 = 1 + \frac{2}{k} + \frac{1}{k^2} < 1 + \frac{1}{2} + \frac{1}{16} < 2.$$

$$2k^2 > \left(\frac{k+1}{k}\right)^2 k^2 = (k+1)^2.$$

3. $S_n = \sum_{1 \le j \le n} \cancel{1} j^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 =$

$$= \frac{n(n+1)(2n+1)}{6}$$

(i) $n=1$ - True

(ii) ~~8~~ Let $S_k = \frac{k(k+1)(2k+1)}{6}$.

$$S_{k+1} = S_k + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 =$$

$$= \frac{k+1}{6} \cdot k(2k+1) + 6(k+1) =$$

$$= \frac{k+1}{6} \cdot 2k^2 + 7k + 6 = \frac{k+1}{6} \cdot (k+2)(2k+3)$$

$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}.$$

4. $S_n = \sum_{1 \le j \le n} (-1)^{j-1} j^2 = 1^2 - 2^2 + 3^2 - \dots + (-1)^{n-1} n^2$

$= (-1)^{n-1} \frac{n(n+1)}{2}$

(i) $n=1$

(ii) Let $S_k = (-1)^{k-1} \frac{k(k+1)}{2}$

$S_k + (-1)^k (k+1)^2 = (-1)^k \left((k+1)^2 - \frac{k(k+1)}{2} \right)$

$= (-1)^k (k+1) \left(\frac{2(k+1) - k}{2} \right) = (-1)^k \frac{(k+1)(k+2)}{2}$

$= S_{k+1}$